Integral-Partial Differential Equations of Isaacs' Type Related to Stochastic Differential Games with Jumps

Rainer Buckdahn*

Département de Mathématiques, Université de Bretagne Occidentale,

6, avenue Victor-le-Gorgeu, CS 93837, 29238 Brest Cedex 3, France.

E-mail: Rainer.Buckdahn@univ-brest.fr.

Ying Hu*

IRMAR, Université Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France.

ying.hu@univ-rennes1.fr.

Juan Li**

School of Mathematics and Statistics, Shandong University at Weihai, Weihai 264200, P. R. China.

E-mail: juanli@sdu.edu.cn.

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Abstract   In this paper we study zero-sum two-player stochastic differential games with jumps with the help of theory of Backward Stochastic Differential Equations (BSDEs). We generalize the results of Fleming and Souganidis [10] and those by Biswas [3] by considering a controlled stochastic system driven by a d-dimensional Brownian motion and a Poisson random measure and by associating nonlinear cost functionals defined by controlled BSDEs. Moreover, unlike the both papers cited above we allow the admissible control processes of both players to depend on all events occurring before the beginning of the game. This quite natural extension allows the players to take into account such earlier events, and it makes even easier to derive the dynamic programming principle. The price to pay is that the cost functionals become random variables and so also the upper and the lower value functions of the game are a priori random fields. The use of a new method allows to prove that, in fact, the upper and the lower value functions are deterministic. On the other hand, the application of BSDE methods [18] allows to prove a dynamic programming principle for the upper and the lower value functions in a very straight-forward way, as well as the fact that they are the unique viscosity solutions of the upper and the lower integral-partial differential equations of Hamilton-Jacobi-Bellman-Isaacs' type, respectively. Finally, the existence of the value of the game is got in this more general setting if Isaacs' condition holds.

1

1 Introduction

In the present work we investigate two-player zero-sum stochastic differential games in the framework of Brownian motion and Poisson random measure. Fleming and Souganidis [10] were the first to study in a rigorous manner two-player zero-sum stochastic differential games (SDGs). They proved that the lower and the upper value functions of such games satisfy the dynamic programming principle (DPP), that they are the unique viscosity solutions of the associated Bellman-Isaacs equations and that they coincide under the Isaacs' condition. Their work has translated former results by Evans and Souganidis [9] from a deterministic into the stochastic framework and has given an important impulse for the research in the theory of stochastic differential games. Various recent works on SDGs are based on the ideas developed in [10], see, for instance, Buckdahn, Cardaliaguet, Rainer [4], Hou, Tang [13], Mataramvura, Øksental [15] and so on, we shall, in particular, also mention the recent work by Biswas [3] on SDGs with jumps. The reader interested in this subject is also referred to the references given in [10].

In the present work we study an extension of the results of the pioneering work of Fleming and Souganidis [10] and those by Biswas [3] on SDGs with jumps. More precisely, inspired by [5], we consider SDGs with jumps on the Wiener-Poisson space and we allow the admissible control processes to depend on the full past of the trajectories of the driving Brownian motion and the Poisson random measure. This means, in particular, that they can also depend on information occurring before the beginning of the game. This approach combined with the notion of stochastic backward semigroups, introduced by Peng [18], simplifies the proof of the DPP considerably. But it also has the consequence that the cost functionals become random variables. In [5], for SDGs in a Brownian setting without jumps, the authors introduced the method of Girsanov transformation, in order to prove that in spite of the randomness of the cost functionals the lower and the upper value functions of the game are deterministic. However, this method doesn't apply to SDGs with jumps. For this reason we study a new type of transformation on the Wiener-Poisson space, which allows to show that also in the case of SDGs with jumps the upper and the lower value functions are deterministic, in spite of controls which can depend on the whole past. Another extension concerns the cost functionals. We consider nonlinear ones, defined through a doubly controlled backward stochastic differential equation (BSDE) with jumps. These both extensions of the framework in [10], [3] and [5] are crucial because they allow to harmonize the setting for stochastic differential games with jumps with that for the stochastic control theory and to simplify considerably the approach in [10] and [3] by using BSDE methods.

BSDEs in the framework of Brownian motion in their general non-linear form were introduced by Pardoux and Peng [17] in 1990. They have been studied since then by a lot of authors and have found various applications, namely in stochastic control, finance and the second order PDE theory. BSDE methods, originally developed by Peng [18] and [19] for the stochastic control theory, have been introduced to the theory of stochastic differential games by Hamadène and Lepeltier [11] and Hamadène, Lepeltier and Peng [12], in order to study games with a dynamics whose diffusion coefficient is strictly elliptic and does not depend on the controls. BSDEs in the framework of Brownian motion and Poisson random measure were first considered by Tang and Li [20], later by Barles, Buckdahn and Pardoux [1], and so on. In Li and Peng [16] they studied the stochastic control theory for BSDE with jumps.

In the present paper we study the general framework of SDGs with jumps. The dynamics of the stochastic differential game in the framework of Brownian motion and compensated Poisson random measure we investigate is given by a doubly controlled system of stochastic differential equations (see equation (3.1)). The cost functionals (interpreted as a payoff for Player I and as a cost for Player II)(see (3.7)) are introduced by a BSDE governed by a Brownian motion and a compensated Poisson random measure (see equation (3.5)). It is well known in the theory of differential games, that players cannot restrict to play only control processes: one player has to fix a strategy while the other player chooses the best answer to this strategy in the form of a control process. So the lower value function W is defined as the essential infimum of the essential supremum of all cost functionals, where the essential infimum is taken over all admissible strategies of Player II and the essential supremum is taken over all admissible controls of Player I. The upper value

function U is defined by changing the roles of the both players: as the essential supremum of the essential infimum of all cost functionals, where the essential supremum is taken over all admissible strategies of Player I and the essential infimum is taken over all admissible controls of Player II; for the precise definitions see (3.9) and (3.10). The objective of our paper is to investigate these lower and upper value functions. The main results of the paper state that W and U are deterministic (Proposition 3.1) continuous unique viscosity solutions of the associated Bellman-Isaacs equations (Theorem 4.1), and they satisfy the DPP (Theorem 3.1).

We point out the fact that W and U, introduced as combination of essential infimum and essential supremum over a class of random variables, are deterministic is far from being trivial. The method developed by Peng [18] (see also Theorem 6.1 of the present paper) for value functions involving only control processes but not strategies does not apply here since the strategies from $\mathcal{A}_{t,T}$ and $\mathcal{B}_{t,T}$ do not have, in general, any continuity property. In [5], the authors used a new method, that of the Girsanov transformation, to solve this difficulty for the stochastic differential games in the framework of Brownian motion, but for the present situation-the SDGs driven by a Brownian motion and a compensated Poisson random measure this method is not applicable anymore. To overcome this difficulty we define a new type of measure-preserving and invertible transformations on the Wiener-Poisson space (see (3.11) and (3.12)). We show in Lemma 3.1 that W and U are invariant under such transformations and in Lemma 3.2 we prove that the invariance of a random variable over the Wiener-Poisson space with respect to these transformations implies that it is deterministic. We emphasize that the proofs of the Lemmas 3.1 and 3.2 do not use BSDE methods. This makes this method also applicable to the other situations, such as standard stochastic control problems with jumps. The importance of the approach which considers control processes depending on events occurring before the beginning of the game, stems from that fact that, once proved that the upper and the lower value functions W and U are deterministic, Peng's notion of backward stochastic semigroups [18] extended to the framework with jumps, allows to prove in a very straight-forward way the DPP and this without any approximation or technical notions (r-strategies and π-controls) playing an essential role in [10] and [3]. Moreover, our approach also allows to show directly with the help of the DPP that W and U are viscosity solutions of the associated Bellman-Isaacs equations.

Our paper is organized as follows. Section 2 recalls some elements of the theory of BSDEs with jumps which will be needed in what follows. Section 3 introduces the setting of stochastic differential game and its lower and upper value functions W and U, and it proves that both these functions are deterministic and satisfy the DPP. The proof of DPP is given in Section 6.2. In Section 4 the DPP allows to derive with the help of Peng's BSDE method [18] adapted to the framework of SDGs with jumps, that W and U are viscosity solutions of the associated Bellman-Isaacs equations. In Section 5 we prove the uniqueness of viscosity solutions of the associated Bellman-Isaacs equations. Finally, after having characterized W and U as viscosity solutions of associated Bellman-Isaacs equations we show that under the Isaacs' condition W and U coincide (one says that the game has a value). Finally, the Appendix recalls some complementary results on FBSDEs with jumps, to which we refer in our work.

2   Preliminaries

Let us begin by introducing the setting for the stochastic differential game we want to investigate. As underlying probability space $(\Omega, \mathcal{F}, P)$ we consider the completed product of the Wiener space $(\Omega_1, \mathcal{F}_1, P_1)$ and the Poisson space $(\Omega_2, \mathcal{F}_2, P_2)$. Here, $(\Omega_1, \mathcal{F}_1, P_1)$ is a Wiener space: Ω_1 is the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}^d$ with value zero at 0, $\Omega_1 = C_0(\mathbb{R}; \mathbb{R}^d)$ endowed with the topology generated by the uniform convergence on compacts. $\mathcal{F}_1$ is the completed Borel σ-algebra over Ω_1, and P_1 the Wiener measure under which the d-dimensional coordinate processes $B_s(\omega) = \omega_s,\ s \in \mathbb{R}_+,\ \omega \in \Omega_1$, and $B_{-s}(\omega) = \omega(-s),\ s \in \mathbb{R}_+,\ \omega \in \Omega_1$, are two independent d-dimensional Brownian motions. By $\{\mathcal{F}_s^B,\ s \geq 0\}$ we denote the natural filtration generated by $\{B_s\}_{s\geq0}$ and augmented by all P_1-null sets, i.e.,

$$\mathcal{F}_s^B = \sigma\{B_r, r \in (-\infty, s]\} \vee \mathcal{N}_{P_1}, s \geq 0.$$

We now introduce the Poisson space $(\Omega_2, \mathcal{F}_2, P_2)$. For this, we let $E = \mathbb{R}^l \setminus \{0\}$ and endow the space E with its Borel σ-field $\mathcal{B}(E)$. By a point function p on E we understand a mapping $p : D_p \subset \mathbb{R} \to E$, where the domain D_p is a countable subset of the real line $\mathbb{R}$. The point function p defines on $\mathbb{R} \times E$ the counting measure $\mu(p, dtde)$ introduced by the relation

$$\mu(p, (s,t] \times \Delta) = \sharp\{r \in D_p \cap (s,t] : p(r) \in \Delta\}, \ \Delta \in \mathcal{B}(E), \ s,t \in \mathbb{R}, \ s < t.$$

In the sequel we will often identify the point function p with $\mu(p,.)$. Let now Ω_2 denote the collection of all point functions p on E and $\mathcal{F}_2$ be the smallest σ-field on Ω_2 with respect to which all mappings $p \to \mu(p, (s,t] \times \Delta), \ s,t \in \mathbb{R}, s < t, \Delta \in \mathcal{B}(E)$ are measurable. On the measurable space $(\Omega_2, \mathcal{F}_2)$ we consider the probability measure P_2 under which the canonical coordinate measure $\mu(p, dtde)$ becomes a Poisson random measure with Lévy measure λ. That means the *compensator* $\hat{\mu}(dtde) = dt\lambda(de)$ of μ transforms $\{\tilde{\mu}((s,t] \times A) = (\mu - \hat{\mu})((s,t] \times A)\}_{s \leq t}$ to a martingale for any $A \in \mathcal{B}(E)$ satisfying $\lambda(A) < \infty$. Here λ is an arbitrarily given σ-finite Lévy measure on $(E, \mathcal{B}(E))$, i.e., a measure on $(E, \mathcal{B}(E))$ with the property that $\int_E (1 \wedge |e|^2)\lambda(de) < \infty$. We complete the probability space $(\Omega_2, \mathcal{F}_2, P_2)$ and introduce the filtration $(\mathcal{F}_t^\mu)_{t \geq 0}$ generated by our coordinate measure μ by setting

$$\dot{\mathcal{F}}_t^\mu = \sigma\{\mu((s,r] \times \Delta) : -\infty < s \leq r \leq t, \Delta \in \mathcal{B}(E)\}, \ t \geq 0,$$

and taking the right-limits $\mathcal{F}_t^\mu = \left(\cap_{\{s>t\}} \dot{\mathcal{F}}_s^\mu\right) \vee \mathcal{N}_{P_2}, \ t \geq 0$, augmented by the P_2-null sets. Finally, we put $\Omega = \Omega_1 \times \Omega_2$, $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2, P = P_1 \otimes P_2$, where $\mathcal{F}$ is completed with respect to P, and the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ is generated by

$$\mathcal{F}_t := \mathcal{F}_t^{B,\mu} = \mathcal{F}_t^B \otimes \mathcal{F}_t^\mu, \ \ t \geq 0, \ \ \text{augmented by all P-null sets.}$$

Let $T > 0$ be an arbitrarily fixed time horizon. For any $n \geq 1$, $|z|$ denotes the Euclidean norm of $z \in \mathbb{R}^n$. We introduce also the following three spaces of processes which will be used frequently in the sequel:

$\mathcal{S}^2(0,T;\mathbb{R}) := \{(\psi_t)_{0 \leq t \leq T}$ real-valued $\mathbb{F}$-adapted càdlàg process :
$$E[\sup_{0 \leq t \leq T} |\psi_t|^2] < +\infty\};$$

$\mathcal{H}^2(0,T;\mathbb{R}^n) := \{(\psi_t)_{0 \leq t \leq T}$ $\mathbb{R}^n$-valued $\mathbb{F}$-progressively measurable process :
$$\| \psi \|^2 = E[\textstyle\int_0^T |\psi_t|^2 dt] < +\infty\};$$

$\mathcal{K}_\lambda^2(0,T;\mathbb{R}^n) := \{$mapping $K : \Omega \times [0,T] \times E \to \mathbb{R}^n \ \mathcal{P} \otimes \mathcal{B}(E)$-measurable :
$$\| K \|^2 = E[\textstyle\int_0^T \int_E |K_t(e)|^2 \lambda(de)dt] < +\infty\}. \ [1]$$

Let us now consider a function $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times L^2(E, \mathcal{B}(E), \lambda; \mathbb{R}) \to \mathbb{R}$ with the property that $(g(t,y,z,k))_{t \in [0,T]}$ is $\mathcal{P}$-measurable for each (y,z,k) in $\mathbb{R} \times \mathbb{R}^d \times L^2(E, \mathcal{B}(E), \lambda; \mathbb{R})$, and we also make the following assumptions on g throughout the paper:

(A1) There exists a constant $C \geq 0$ such that, P-a.s., for all $t \in [0,T]$, $y_1, y_2 \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^d$, $k_1, k_2 \in L^2(E, \mathcal{B}(E), \lambda; \mathbb{R})$,
$$|g(t,y_1,z_1,k_1) - g(t,y_2,z_2,k_2)| \leq C(|y_1 - y_2| + |z_1 - z_2| + ||k_1 - k_2||).$$

(A2) $g(\cdot, 0, 0, 0) \in \mathcal{H}^2(0,T;\mathbb{R})$.

The following result on BSDEs with jumps is by now well known, for its proof the reader is referred to Lemma 2.4 in Tang and Li [20] or Theorem 2.1 in Buckdahn, Barles and Pardoux [1].

Lemma 2.1. *Under the assumptions (A1) and (A2), for any random variable $\xi \in L^2(\emptyset, \mathcal{F}_T, P)$, the BSDE with jump*

$$y_t = \xi + \int_t^T g(s, y_s, z_s, k_s)ds - \int_t^T z_s \, dB_s - \int_t^T \int_E k_s(e)\tilde{\mu}(ds, de), \quad 0 \leq t \leq T, \tag{2.1}$$

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[1] $\mathcal{P}$ denotes the σ-algebra of $\mathcal{F}_t$-predictable subsets of $\Omega \times [0,T]$.

has a unique adapted solution

$$(y_t^{T,g,\xi}, z_t^{T,g,\xi}, k_t^{T,g,\xi})_{t\in[0,T]} \in \mathcal{S}^2(0,T;\mathbb{R}) \times \mathcal{H}^2(0,T;\mathbb{R}^d) \times \mathcal{K}_\lambda^2(0,T;\mathbb{R}).$$

In the sequel, we always assume that the driving coefficient g of a BSDE with jump satisfies (A1) and (A2).

We recall also the following both basic results on BSDEs with jumps. We begin with the well-known comparison theorem (see Barles, Buckdahn and Pardoux [1], Proposition 2.6).

Lemma 2.2. *(Comparison Theorem) Let $h : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$ be $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R})$ measurable and satisfy*

(i) *There exists a constant $C \geq 0$ such that, P-a.s., for all $t \in [0,T]$, $y_1, y_2 \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^d$, $k_1, k_2 \in \mathbb{R}$,*

$$|h(t, y_1, z_1, k_1) - h(t, y_2, z_2, k_2)| \leq C(|y_1 - y_2| + |z_1 - z_2| + |k_1 - k_2|).$$

(ii) *$h(\cdot, 0, 0, 0) \in \mathcal{H}^2(0,T;\mathbb{R})$.*

(iii) *$k \to h(t, y, z, k)$ is non-decreasing, for all $(t, y, z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$.*

Furthermore, let $l : \Omega \times [0,T] \times E \to \mathbb{R}$ be $\mathcal{P} \otimes \mathcal{B}(E)$ measurable and satisfy

$$0 \leq l_t(e) \leq C(1 \wedge |e|), \quad e \in E.$$

We set

$$g(t, \omega, y, z, \varphi) = h(t, \omega, y, z, \int_E \varphi(e)l_t(\omega, e)\lambda(de)),$$

for $(t, \omega, y, z, \varphi) \in [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times L^2(E, \mathcal{B}(E), \lambda; \mathbb{R})$.

Let ξ', $\xi \in L^2(\emptyset, \mathcal{F}_T, P)$ and g' satisfies (A1) and (A2).

We denote by (y, z, k) (resp., (y', z', k')) the unique solution of equation (2.1) with the data (ξ, g) (resp., (ξ', g')). If

(iv) *$\xi \geq \xi'$, a.s.;*

(v) *$g(t, y, z, k) \geq g'(t, y, z, k)$, a.s., a.e., for any $(y, z, k) \in \mathbb{R} \times \mathbb{R}^d \times L^2(E, \mathcal{B}(E), \lambda; \mathbb{R})$.*

Then, we have: $y_t \geq y_t'$, a.s., for all $t \in [0,T]$. And if, in addition, we also assume that $P(\xi_1 > \xi_2) > 0$, then $P\{y_t > y_t'\} > 0$, $0 \leq t \leq T$, and in particular, $y_0 > y_0'$.

Using the notation introduced in Lemma 2.1 we now suppose that, for some $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times L^2(E, \mathcal{B}(E), \lambda; \mathbb{R}) \longrightarrow \mathbb{R}$ satisfying (A1) and (A2) and for some $i \in \{1, 2\}$, the drivers g_i are of the form

$$g_i(s, y_s^i, z_s^i, k_s^i) = g(s, y_s^i, z_s^i, k_s^i) + \varphi_i(s), \quad \text{dsdP-a.e.,}$$

where $\varphi_i \in \mathcal{H}^2(0,T;\mathbb{R})$. Then, for terminal values ξ_1, ξ_2 belonging to $L^2(\Omega, \mathcal{F}_T, P)$ we have the following

Lemma 2.3. *The difference of the solutions (y^1, z^1, k^1) and (y^2, z^2, k^2) of BSDE (2.1) with the data (ξ_1, g_1) and (ξ_2, g_2), respectively, satisfies the following estimate:*

$$
\begin{aligned}
&|y_t^1 - y_t^2|^2 + \tfrac{1}{2}E[\int_t^T e^{\beta(s-t)}(|y_s^1 - y_s^2|^2 + |z_s^1 - z_s^2|^2)ds|\mathcal{F}_t] \\
&\quad + \tfrac{1}{2}E[\int_t^T \int_E e^{\beta(s-t)}|k_s^1(e) - k_s^2(e)|^2\lambda(de)ds|\mathcal{F}_t] \\
\leq\ & E[e^{\beta(T-t)}|\xi_1 - \xi_2|^2|\mathcal{F}_t] + E[\int_t^T e^{\beta(s-t)}|\varphi_1(s) - \varphi_2(s)|^2ds|\mathcal{F}_t], \ P\text{-a.s., for all } 0 \leq t \leq T,
\end{aligned}
$$

where $\beta \geq 2 + 2C + 4C^2$.

For the proof the reader is referred to Barles, Buckdahn and Pardoux [1], Proposition 2.2.

3 A DPP for stochastic differential games with jumps

Now we begin to consider the stochastic differential games with jumps under our setting.

The set of admissible control processes $\mathcal{U}$ (resp., $\mathcal{V}$) for the first (resp., second) player is the set of all U (resp., V)-valued $\mathcal{F}_t$-predictable processes. The control state spaces U and V are supposed to be compact metric spaces.

For given admissible controls $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$, the corresponding orbit which regards t as the initial time and $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$ as the initial state is defined by the solution of the following SDE with jump:

$$\begin{cases} dX_s^{t,\zeta;u,v} & = & b(s, X_s^{t,\zeta;u,v}, u_s, v_s)ds + \sigma(s, X_s^{t,\zeta;u,v}, u_s, v_s)dB_s \\ & & + \int_E \gamma(s, X_{s-}^{t,\zeta;u,v}, u_s, v_s, e)\widetilde{\mu}(ds, de), \qquad s \in [t, T], \\ X_t^{t,\zeta;u,v} & = & \zeta, \end{cases} \tag{3.1}$$

where the mappings

$$b : [0, T] \times \mathbb{R}^n \times U \times V \to \mathbb{R}^n, \qquad \sigma : [0, T] \times \mathbb{R}^n \times U \times V \to \mathbb{R}^{n \times d},$$
$$\gamma : [0, T] \times \mathbb{R}^n \times U \times V \times E \to \mathbb{R}^n,$$

satisfy the following conditions:

(i) For every fixed $(x, e) \in \mathbb{R}^n \times E$, $b(., x, ., .), \sigma(., x, ., .)$ and $\gamma(., x, ., ., e)$ are continuous in (t, u, v);

(ii) There exists a constant $C > 0$ such that, for all $t \in [0, T]$, $x, x' \in \mathbb{R}^n$, $u \in U$, $v \in V$,
$$|b(t, x, u, v) - b(t, x', u, v)| + |\sigma(t, x, u, v) - \sigma(t, x', u, v)| \le C|x - x'|.$$

(iii) There exists $\rho : E \to \mathbb{R}^+$ with $\int_E \rho^2(e)\lambda(de) < +\infty$, such that, for any $t \in [0, T], x, y \in \mathbb{R}^n, u \in U, v \in V$ and $e \in E$,
$$|\gamma(t, x, u, v, e) - \gamma(t, y, u, v, e)| \le \rho(e)|x - y|,$$
$$|\gamma(t, 0, u, v, e)| \le \rho(e).$$

(H3.1)

From (H3.1) we get the global linear growth conditions of b and σ, i.e., the existence of some $C > 0$ such that, for all $0 \le t \le T$, $u \in U$, $v \in V$, $x \in \mathbb{R}^n$,

$$\begin{aligned} |b(t, x, u, v)| + |\sigma(t, x, u, v)| & \le & C(1 + |x|); \\ |\gamma(t, x, u, v, e)| & \le & \rho(e)(1 + |x|). \end{aligned} \tag{3.2}$$

Obviously, under the above assumptions, for any $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$, SDE (3.1) has a unique strong solution. Moreover, there exists $C \in \mathbb{R}^+$ such that, for any $t \in [0, T]$, $u(\cdot) \in \mathcal{U}, v(\cdot) \in \mathcal{V}$ and $\zeta, \zeta' \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, we have the following estimates, P-a.s.:

$$\begin{aligned} E[\sup_{s \in [t, T]} |X_s^{t,\zeta;u,v} - X_s^{t,\zeta';u,v}|^2 | \mathcal{F}_t] & \le & C|\zeta - \zeta'|^2, \\ E[\sup_{s \in [t, T]} |X_s^{t,\zeta;u,v}|^2 | \mathcal{F}_t] & \le & C(1 + |\zeta|^2). \end{aligned} \tag{3.3}$$

The constant C depends only on the Lipschitz and the linear growth constants of b, σ and γ with respect to x.

Let now be given three measurable functions

$$\Phi : \mathbb{R}^n \to \mathbb{R}, \ f : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times U \times V \to \mathbb{R}, \ l : \mathbb{R}^n \times E \to \mathbb{R}$$

which satisfy the following conditions:

(i) For every fixed $(x, y, z, k) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$, $f(., x, y, z, k, ., .)$ is continuous in (t, u, v) and there exists a constant $C > 0$ such that, for all $t \in [0, T]$, $x, x' \in \mathbb{R}^n$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$, $k, k' \in \mathbb{R}$, $u \in U$ and $v \in V$,

$$|f(t, x, y, z, k, u, v) - f(t, x', y', z', k', u, v)|$$
$$\leq C(|x - x'| + |y - y'| + |z - z'| + |k - k'|);$$

(ii) $k \to f(t, x, y, z, k, u, v)$ is non-decreasing, for all $(t, x, y, z, u, v) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times U \times V$;   (H3.2)

(iii) There exists a constant $C > 0$ such that,
$$0 \leq l(x, e) \leq C(1 \wedge |e|), \quad x \in \mathbb{R}^n, \ e \in E,$$
$$|l(x, e) - l(x', e)| \leq C|x - x'|(1 \wedge |e|), \ x, x' \in \mathbb{R}^n, \ e \in E;$$

(iv) There exists a constant $C > 0$ such that, for all $x, x' \in \mathbb{R}^n$,
$$|\Phi(x) - \Phi(x')| \leq C|x - x'|.$$

From (H3.2) we see that f and Φ also satisfy the global linear growth condition in x, i.e., there exists some $C > 0$ such that, for all $0 \leq t \leq T$, $u \in U$, $v \in V$, $x \in \mathbb{R}^n$,

$$|f(t, x, 0, 0, 0, u, v)| + |\Phi(x)| \leq C(1 + |x|). \tag{3.4}$$

For any $u(\cdot) \in \mathcal{U}$, $v(\cdot) \in \mathcal{V}$ and $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, the mappings $\xi := \Phi(X_T^{t,\zeta;u,v})$ and $g(s, y, z, k) := f(s, X_s^{t,\zeta;u,v}, y, z, \int_E k(e)l(X_s^{t,\zeta;u,v}, e)\lambda(de), u_s, v_s)$, $(s, y, z, k) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times L^2(E, \mathcal{B}(E), \lambda; \mathbb{R})$ satisfy the conditions of Lemma 2.1 on the interval $[t, T]$. Therefore, there exists a unique solution to the following BSDE:

$$\begin{cases} -dY_s^{t,\zeta;u,v} = f(s, X_s^{t,\zeta;u,v}, Y_s^{t,\zeta;u,v}, Z_s^{t,\zeta;u,v}, \int_E K_s^{t,\zeta;u,v}(e)l(X_s^{t,\zeta;u,v}, e)\lambda(de), u_s, v_s)ds \\ \qquad\qquad -Z_s^{t,\zeta;u,v}dB_s - \int_E K_s^{t,\zeta;u,v}(e)\widetilde{\mu}(ds, de), \\ Y_T^{t,\zeta;u,v} = \Phi(X_T^{t,\zeta;u,v}), \end{cases} \tag{3.5}$$

where $X^{t,\zeta;u,v}$ is introduced by equation (3.1).

Note that in (3.5) and in the sequel, f depends on K in a very specific way in order to make full use of the comparison theorem-Lemma 2.2.

Moreover, in analogy to Proposition 6.1 in the Appendix, we can see that there exists some constant $C > 0$ such that, for all $0 \leq t \leq T$, $\zeta, \zeta' \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$, P-a.s.,

(i) $|Y_t^{t,\zeta;u,v} - Y_t^{t,\zeta';u,v}| \leq C|\zeta - \zeta'|$;

(ii) $|Y_t^{t,\zeta;u,v}| \leq C(1 + |\zeta|)$.   (3.6)

Now, similar to [5] and [10], we introduce the following subspaces of admissible controls and the definition of admissible strategies for the game:

Definition 3.1. *An admissible control process $u = \{u_r, r \in [t, s]\}$ (resp., $v = \{v_r, r \in [t, s]\}$) for Player I (resp., II) on $[t, s](t < s \leq T)$ is an $\mathcal{F}_r$-predictable process taking values in U (resp., V). The set of all admissible controls for Player I (resp., II) on $[t, s]$ is denoted by $\mathcal{U}_{t,s}$ (resp., $\mathcal{V}_{t,s}$). We identify both processes u and $\bar{u}$ in $\mathcal{U}_{t,s}$ and write $u \equiv \bar{u}$ on $[t, s]$, if $P\{u = \bar{u} \ a.e. \ in \ [t, s]\} = 1$. Similarly we interpret $v \equiv \bar{v}$ on $[t, s]$ in $\mathcal{V}_{t,s}$.*

Finally, we still have to define the admissible strategies for the game.

Definition 3.2. *A nonanticipative strategy for Player I on $[t, s](t < s \leq T)$ is a mapping $\alpha : \mathcal{V}_{t,s} \longrightarrow \mathcal{U}_{t,s}$ such that, for any $\mathcal{F}_r$-stopping time $S : \Omega \to [t, s]$ and any $v_1, v_2 \in \mathcal{V}_{t,s}$ with $v_1 \equiv v_2$ on $[\![t, S]\!]$, it holds $\alpha(v_1) \equiv \alpha(v_2)$ on $[\![t, S]\!]$. Nonanticipative strategies for Player II on $[t, s]$, $\beta : \mathcal{U}_{t,s} \longrightarrow \mathcal{V}_{t,s}$, are defined similarly. The set of all nonanticipative strategies $\alpha : \mathcal{V}_{t,s} \longrightarrow \mathcal{U}_{t,s}$ for Player I on $[t, s]$ is denoted by $\mathcal{A}_{t,s}$. The set of all nonanticipative strategies $\beta : \mathcal{U}_{t,s} \longrightarrow \mathcal{V}_{t,s}$ for Player II on $[t, s]$ is denoted by $\mathcal{B}_{t,s}$. (Recall that $[\![t, S]\!] = \{(r, \omega) \in [0, T] \times \Omega, t \leq r \leq S(\omega)\}$.)*

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Given the control processes $u(\cdot) \in \mathcal{U}_{t,T}$ and $v(\cdot) \in \mathcal{V}_{t,T}$ we introduce the following associated cost functional

$$J(t, x; u, v) := Y_t^{t,x;u,v}, \ (t, x) \in [0, T] \times \mathbb{R}^n, \tag{3.7}$$

where the process $Y^{t,x;u,v}$ is defined by BSDE (3.5).
Similarly to the proof of Theorem 6.1 in the Appendix, we can get that, for any $t \in [0, T]$ and $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$,

$$J(t, \zeta; u, v) = Y_t^{t,\zeta;u,v}, \ \text{P-a.s.} \tag{3.8}$$

Being particularly interested in the case of a deterministic ζ, i.e., $\zeta = x \in \mathbb{R}^n$, we define the lower value function of our stochastic differential game

$$W(t, x) := \operatorname{essinf}_{\beta \in \mathcal{B}_{t,T}} \operatorname{esssup}_{u \in \mathcal{U}_{t,T}} J(t, x; u, \beta(u)) \tag{3.9}$$

and its upper value function

$$U(t, x) := \operatorname{esssup}_{\alpha \in \mathcal{A}_{t,T}} \operatorname{essinf}_{v \in \mathcal{V}_{t,T}} J(t, x; \alpha(v), v). \tag{3.10}$$

Remark 3.1. *(1) Here the essential infimum and the essential supremum should be understood as one with respect to indexed families of random variables (see, e.g., Dunford and Schwartz [8], Dellacherie [7] or the Appendix in Karatzas and Shreve [14] for detailed discussions). The reader is also referred to Remark 3.1 in [5].*
(2) Let us point out that under our conditions (H3.1)-(H3.2) the lower value function $W(t, x)$ and the upper value function $U(t, x)$ are well defined and, a priorily, bounded, $\mathcal{F}_t$-measurable random variables. However, we show below that they are indeed deterministic functions. Such a result was already got in the case of stochastic differential games only driven by a Brownian motion (see [5]). However, here, in presence of an additional driving compensated Poisson random measure, the argument of the Girsanov transformation employed in [5] doesn't work anymore and has to be replaced by a quite different transformation argument. In what follows we concentrate on the study of W, the upper value function U can be investigated in a similar manner.

Proposition 3.1. *For any $(t, x) \in [0, T] \times \mathbb{R}^n$, we have $W(t, x) = E[W(t, x)]$, P-a.s. Thus, let $W(t, x)$ identify with its deterministic version $E[W(t, x)]$, $W : [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is a deterministic function.*

The proof will be split into two lemmas.

Lemma 3.1. *Let $(t, x) \in [0, T] \times \mathbb{R}^n$ and $\tau : \Omega \to \Omega$ be an invertible $\mathcal{F}$-$\mathcal{F}$ measurable transformation such that*

      i) *τ and $\tau^{-1} : \Omega \to \Omega$ are $\mathcal{F}_t - \mathcal{F}_t$ measurable;*
      ii) *$(B_s - B_t) \circ \tau = B_s - B_t$, $s \in [t, T]$;*
         $\mu((t, s] \times A) \circ \tau = \mu((t, s] \times A)$, $s \in [t, T]$, $A \in \mathcal{B}(E)$;
      iii) *the law $P \circ [\tau]^{-1}$ of τ is equivalent to the underlying probability measure P.*

Then, $W(t, x) \circ \tau = W(t, x)$, P-a.s.

Proof: We split now the proof in the following steps:
1st step: For any $u \in \mathcal{U}_{t,T}$, $v \in \mathcal{V}_{t,T}$, $J(t, x; u, v) \circ \tau = J(t, x; u(\tau), v(\tau))$, P-a.s.

Indeed, we apply the transformation τ to SDE (3.1) (with $\zeta = x$) and compare the obtained equation with the SDE obtained from (3.1) by substituting the controlled processes $u(\tau)$, $v(\tau)$ for u and v. Then, from the uniqueness of the solution of (3.1) we get $X_s^{t,x;u,v}(\tau) = X_s^{t,x;u(\tau),v(\tau)}$, for any $s \in [t, T]$, P-a.s. Furthermore, by a similar transformation argument we obtain from the uniqueness of the solution of BSDE (3.5),

$$Y_s^{t,x;u,v}(\tau) = Y_s^{t,x;u(\tau),v(\tau)}, \ \text{for any } s \in [t, T], \ \text{P-a.s.},$$

$$Z_s^{t,x;u,v}(\tau) = Z_s^{t,x;u(\tau),v(\tau)}, \ \text{dsdP-a.e. on } [t, T] \times \Omega,$$

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$$K_s^{t,x;u,v}(\tau) = K_s^{t,x;u(\tau),v(\tau)}, \ ds\lambda(de)\mathrm{dP}\text{-a.e. on } [t,T] \times E \times \Omega.$$

Consequently, in particular, we have

$$J(t,x;u,v)(\tau) = J(t,x;u(\tau),v(\tau)), \ \mathrm{P\text{-}a.s.}$$

2nd step: For $\beta \in \mathcal{B}_{t,T}$, let $\widehat{\beta}(u) := \beta(u(\tau^{-1}))(\tau)$, $u \in \mathcal{U}_{t,T}$. Then, $\widehat{\beta} \in \mathcal{B}_{t,T}$.

Obviously, $\widehat{\beta}$ maps $\mathcal{U}_{t,T}$ into $\mathcal{V}_{t,T}$. Moreover, this mapping $\widehat{\beta}$ is nonanticipating. Indeed, let $S : \Omega \to [t,T]$ be an $\mathbb{F}$-stopping time and $u_1, u_2 \in \mathcal{U}_{t,T}$ such that $u_1 \equiv u_2$ on $[\![t, \ S]\!]$. Then, obviously, $u_1(\tau^{-1}) \equiv u_2(\tau^{-1})$ on $[\![t, \ S(\tau^{-1})]\!]$ (notice that $S(\tau^{-1})$ is still an $\mathbb{F}$-stopping time. For this we use that the assumptions i) and ii) imply that $\tau(\mathcal{F}_s) := \{\tau(A), \ A \in \mathcal{F}_s\} = \mathcal{F}_s, \ s \in [t,T]$). Thus, because $\beta \in \mathcal{B}_{t,T}$, we have $\beta(u_1(\tau^{-1})) \equiv \beta(u_2(\tau^{-1}))$ on $[\![t, \ S(\tau^{-1})]\!]$. Therefore,

$$\widehat{\beta}(u_1) = \beta(u_1(\tau^{-1}))(\tau) \equiv \beta(u_2(\tau^{-1}))(\tau) = \widehat{\beta}(u_2) \text{ on } [\![t, \ S]\!].$$

3rd step: For all $\beta \in \mathcal{B}_{t,T}$ we have:

$$(\mathrm{esssup}_{u \in \mathcal{U}_{t,T}} J(t,x;u,\beta(u)))(\tau) = \mathrm{esssup}_{u \in \mathcal{U}_{t,T}}(J(t,x;u,\beta(u))(\tau)), \ \mathrm{P\text{-}a.s..}$$

Indeed, with the notation $I(t,x;\beta) := \mathrm{esssup}_{u \in \mathcal{U}_{t,T}} J(t,x;u,\beta(u))$, $\beta \in \mathcal{B}_{t,T}$, we have $I(t,x;\beta) \geq J(t,x;u,\beta(u))$, and thus $I(t,x;\beta)(\tau) \geq J(t,x;u,\beta(u))(\tau)$, P-a.s., for all $u \in \mathcal{U}_{t,T}$ (recall that $P \circ \tau^{-1}$ is equivalent to P due to assumption iii)). On the other hand, for any random variable ζ satisfying $\zeta \geq J(t,x;u,\beta(u))(\tau)$, and hence also $\zeta(\tau^{-1}) \geq J(t,x;u,\beta(u))$, P-a.s., for all $u \in \mathcal{U}_{t,T}$, we have $\zeta(\tau^{-1}) \geq I(t,x;\beta)$, P-a.s., i.e., $\zeta \geq I(t,x;\beta)(\tau)$, P-a.s. Consequently,

$$I(t,x;\beta)(\tau) = \mathrm{esssup}_{u \in \mathcal{U}_{t,T}}(J(t,x;u,\beta(u))(\tau)), \ \mathrm{P\text{-}a.s.}$$

4th step: $W(t,x)$ is invariant with respect to the transformation τ, i.e.,

$$W(t,x)(\tau) = W(t,x), \ \mathrm{P\text{-}a.s.}$$

Indeed, similarly to the third step we can show that:

$$(\mathrm{essinf}_{\beta \in \mathcal{B}_{t,T}} I(t,x;\beta))(\tau) = \mathrm{essinf}_{\beta \in \mathcal{B}_{t,T}}(I(t,x;\beta)(\tau)), \ \mathrm{P\text{-}a.s.}$$

Then, from the first step to the third step we have,

$$
\begin{aligned}
W(t,x)(\tau) &= \mathrm{essinf}_{\beta \in \mathcal{B}_{t,T}} \mathrm{esssup}_{u \in \mathcal{U}_{t,T}} (J(t,x;u,\beta(u))(\tau)) \\
&= \mathrm{essinf}_{\beta \in \mathcal{B}_{t,T}} \mathrm{esssup}_{u \in \mathcal{U}_{t,T}} J(t,x;u(\tau),\widehat{\beta}(u(\tau))) \\
&= \mathrm{essinf}_{\beta \in \mathcal{B}_{t,T}} \mathrm{esssup}_{u \in \mathcal{U}_{t,T}} J(t,x;u,\widehat{\beta}(u)) \\
&= \mathrm{essinf}_{\beta \in \mathcal{B}_{t,T}} \mathrm{esssup}_{u \in \mathcal{U}_{t,T}} J(t,x;u,\beta(u)) \\
&= W(t,x), \ \mathrm{P\text{-}a.s.,}
\end{aligned}
$$

where we have used $\{u(\tau) \mid u(\cdot) \in \mathcal{U}_{t,T}\} = \mathcal{U}_{t,T}$, $\{\widehat{\beta} \mid \beta \in \mathcal{B}_{t,T}\} = \mathcal{B}_{t,T}$ in order to obtain the both latter equalities. $\qquad\square$

Now let $\ell \geq 1$. We define the transformation $\tau'_\ell : \Omega_1 \to \Omega_1$ such that, for all $\omega_1 \in \Omega_1 = C_0(\mathbb{R}; \mathbb{R}^d)$,

$$
\begin{aligned}
&(\tau'_\ell \omega_1)((t-\ell, r]) = \omega_1((t-2\ell, r-\ell])(:= \omega_1(r-\ell) - \omega_1(t-2\ell)); \\
&(\tau'_\ell \omega_1)((t-2\ell, r-\ell]) = \omega_1((t-\ell, r]), \text{ for } r \in [t-\ell, t]; \\
&(\tau'_\ell \omega_1)((s, r]) = \omega_1((s, r]), \ (s, r] \cap (t-2\ell, t] = \emptyset; \\
&(\tau'_\ell \omega_1)(0) = 0.
\end{aligned}
\tag{3.11}
$$

Moreover, for $p \in \Omega_2$, $p = \Sigma_{x \in D_p} p(x) \delta_x$, we put:

$$\tau''_\ell p := \Sigma_{x \in D_p \cap (t-2\ell, t]^c} p(x) \delta_x + \Sigma_{x \in D_p \cap (t-\ell, t]} p(x) \delta_{x-\ell} + \Sigma_{x \in D_p \cap (t-2\ell, t-\ell]} p(x) \delta_{x+\ell}.$$

Obviously, $\tau_\ell'' : \Omega_2 \to \Omega_2$ is a bijection, $\tau_\ell''^{-1} = \tau_\ell''$, which preserves the measure $P_2 \circ [\tau_\ell'']^{-1} = P_2$. Moreover,

$$
\begin{aligned}
\mu(\tau_\ell'' p; \ (t-\ell, r] \times \Delta) &= \mu(p; \ (t-2\ell, r-\ell] \times \Delta), \ r \in (t-\ell, t], \ \Delta \in \mathcal{B}(E); \\
\mu(\tau_\ell'' p; \ (t-2\ell, r-\ell] \times \Delta) &= \mu(p; \ (t-\ell, r] \times \Delta), \ r \in (t-\ell, t], \ \Delta \in \mathcal{B}(E); \\
\mu(\tau_\ell'' p; \ (s,r] \times \Delta) &= \mu(p; \ (s,r] \times \Delta), \ (s,r] \cap (t-2\ell, t] = \emptyset, \ \Delta \in \mathcal{B}(E).
\end{aligned} \tag{3.12}
$$

Thus, the transformation $\tau_\ell : \Omega \to \Omega$, $\tau_\ell \omega := (\tau_\ell' \omega_1, \ \tau_\ell'' p)$, $\omega = (\omega_1, \ p) \in \Omega = \Omega_1 \times \Omega_2$, satisfies the assumptions i), ii), iii) of Lemma 3.1. Therefore, $W(t,x)(\tau_\ell) = W(t,x)$, P-a.s., $\ell \geq 1$. The proof of Proposition 3.1 will be completed by the following auxiliary Lemma 3.2.

Lemma 3.2. *Let $\zeta \in L^\infty(\Omega, \mathcal{F}_t, P)$ be such that, for all $\ell \geq 1$ natural number, $\zeta(\tau_\ell) = \zeta$, P-a.e. Then, there exists some real C such that $\zeta = C$, P-a.s.*

Proof: To simplify the notation we introduce the Brownian motion $B_r' := B_t - B_{t-r}$, $r \geq 0$, and the Poisson random measure

$$
\upsilon([0,r] \times \Delta) := \mu([t-r, t] \times \Delta), \ \ r \geq 0, \ \Delta \in \mathcal{B}(E),
$$

on $\mathbb{R}_+ \times E$. Then,

$$
\begin{aligned}
B_r'(\tau_\ell' \omega_1) &= B_{\ell+r}'(\omega_1) - B_\ell'(\omega_1); \\
(B_{\ell+r}' - B_\ell')(\tau_\ell' \omega_1) &= B_r'(\omega_1), \ r \in [0, \ell]; \\
B_r'(\tau_\ell' \omega_1) - B_s'(\tau_\ell' \omega_1) &= B_r'(\omega_1) - B_s'(\omega_1), \ (s,r] \cap [0, 2\ell] = \emptyset;
\end{aligned}
$$

and

$$
\begin{aligned}
\upsilon(\tau_\ell'' p; \ [0,r] \times \Delta) &= \upsilon(p; \ [\ell, \ell+r] \times \Delta); \\
\upsilon(\tau_\ell'' p; \ [\ell, \ell+r] \times \Delta) &= \upsilon(p; \ [0,r] \times \Delta), \ r \in [0, \ell], \ \Delta \in \mathcal{B}(E); \\
\upsilon(\tau_\ell'' p; \ [s,r] \times \Delta) &= \upsilon(p; \ [s,r] \times \Delta), \ [s,r] \cap [0, 2\ell] = \emptyset, \ 0 \leq s \leq r, \ \Delta \in \mathcal{B}(E).
\end{aligned}
$$

Moreover, we put $\mathcal{F}_{s,r}' := \sigma\{B_{s'}' - B_s', \ \upsilon([s,s'] \times \Delta), \ s' \in [s,r], \ \Delta \in \mathcal{B}(E)\} \vee \mathcal{N}_P, 0 \leq s \leq r < +\infty$. Let $\zeta \in L^\infty(\Omega, \mathcal{F}_t, P)$ be such that $\zeta(\tau_\ell) = \zeta$, P-a.s., for all $\ell \geq 1$ natural number. To prove Lemma 3.2 it suffices to show that $\zeta = E[\zeta]$, P-a.s.

To this end we consider a random variable of the form $\theta \eta \varsigma$, where $\theta \in L^\infty(\Omega, \mathcal{F}_{0,\ell}', P)$, $\eta \in L^\infty(\Omega, \mathcal{F}_{\ell,2\ell}', P)$, and $\varsigma \in L^\infty(\Omega, \mathcal{F}_{2\ell,\infty}', P)$. Then, $\theta(\tau_\ell) \in L^\infty(\Omega, \mathcal{F}_{\ell,2\ell}', P)$, and the random variable $\eta(\tau_\ell) \in L^\infty(\Omega, \mathcal{F}_{0,\ell}', P)$ is independent of $\mathcal{F}_{\ell,\infty}'$. Consequently, taking into account that $\varsigma(\tau_\ell) = \varsigma$, P-a.e., we have

$$
E[(\theta\eta\varsigma)(\tau_\ell)|\mathcal{F}_{\ell,\infty}'] = E[\eta(\tau_\ell)]\theta(\tau_\ell)\varsigma = E[\eta]\theta(\tau_\ell)\varsigma = E[\theta\eta\varsigma|\mathcal{F}_{0,\ell}' \vee \mathcal{F}_{2\ell,\infty}'](\tau_\ell), \ \text{P-a.s.},
$$

and from the monotone class theorem we conclude that, for all $\theta \in L^1(\Omega, \mathcal{F}_\infty', P)$,

$$
E[\theta(\tau_\ell)|\mathcal{F}_{\ell,\infty}'] = E[\theta|\mathcal{F}_{0,\ell}' \vee \mathcal{F}_{2\ell,\infty}'](\tau_\ell), \ \text{P-a.s.}, \ \ell \geq 1.
$$

Thus, $E[\zeta|\mathcal{F}_{\ell,\infty}'] = E[\zeta(\tau_\ell)|\mathcal{F}_{\ell,\infty}'] = E[\zeta|\mathcal{F}_{0,\ell}' \vee \mathcal{F}_{2\ell,\infty}'](\tau_\ell)$, P-a.s., $\ell \geq 1$.

As $E[\zeta|\mathcal{F}_{\ell,\infty}'] \to E[\zeta | \cap_{\ell \geq 1} \mathcal{F}_{\ell,\infty}'] = E[\zeta]$, as $\ell \to \infty$, P-a.s., and in L^1, it follows that

$$
\begin{aligned}
&E[|E[\zeta|\mathcal{F}_{0,\ell}'] - E[\zeta]|] \\
&= E[|E[E[\zeta|\mathcal{F}_{0,\ell}' \vee \mathcal{F}_{2\ell,\infty}'] - E[\zeta]|\mathcal{F}_{0,\ell}']|] \\
&\leq E[|E[\zeta|\mathcal{F}_{0,\ell}' \vee \mathcal{F}_{2\ell,\infty}'] - E[\zeta]|] \\
&= E[|E[\zeta|\mathcal{F}_{0,\ell}' \vee \mathcal{F}_{2\ell,\infty}'](\tau_\ell) - E[\zeta]|] \quad (\text{Recall}: P \circ [\tau_\ell]^{-1} = P) \\
&= E[|E[\zeta|\mathcal{F}_{\ell,\infty}'] - E[\zeta]|] \to 0, \ \text{as } \ell \to +\infty.
\end{aligned}
$$

Consequently, $E[\zeta|\mathcal{F}_{0,\ell}'] \to E[\zeta]$ in L^1, as $\ell \to \infty$. But, on the other hand, $E[\zeta|\mathcal{F}_{0,\ell}'] \to E[\zeta|\mathcal{F}_{0,\infty}'] = \zeta$, in L^1, as $\ell \to \infty$. This shows that $\zeta = E[\zeta]$, P-a.s. $\qquad \blacksquare$

Remark 3.2. *From Lemma 3.2 we know $W(t,x)$ is independent of $\mathcal{F}_T$.*

The first property of the lower value function $W(t, x)$ which we present is an immediate consequence of its definition (3.9) and the Lipschitz property (3.6) of the cost functionals.

Lemma 3.3. *There exists a constant $C > 0$ such that, for all $0 \le t \le T$, $x, x' \in \mathbb{R}^n$,*

$$
\begin{array}{ll}
\text{(i)} & |W(t, x) - W(t, x')| \le C|x - x'|; \\
\text{(ii)} & |W(t, x)| \le C(1 + |x|).
\end{array}
\tag{3.13}
$$

$\square$

We now discuss (the generalized) dynamic programming principle (DPP) for our stochastic differential game (3.1), (3.5) and (3.9). For this end we have to define the family of (backward) semigroups associated with BSDE (3.5). This notion of the stochastic backward semigroup was first introduced by Peng [18] which was applied to study the DPP for stochastic control problems in the framework of Brownian motion. Our approach adapts Peng's ideas to the framework of stochastic differential games with jumps.

Given the initial data (t, x), a positive number $\delta \le T - t$, admissible control processes $u(\cdot) \in \mathcal{U}_{t,t+\delta}$, $v(\cdot) \in \mathcal{V}_{t,t+\delta}$ and a real-valued random variable $\eta \in L^2(\Omega, \mathcal{F}_{t+\delta}, P; \mathbb{R})$, we put

$$
G_{s,t+\delta}^{t,x;u,v}[\eta] := \tilde{Y}_s^{t,x;u,v}, \qquad s \in [t, t+\delta],
\tag{3.14}
$$

where $(\tilde{Y}_s^{t,x;u,v}, \tilde{Z}_s^{t,x;u,v}, \widetilde{K}_s^{t,x;u,v})_{t \le s \le t+\delta}$ is the solution of the following BSDE with the time horizon $t + \delta$:

$$
\begin{cases}
-d\tilde{Y}_s^{t,x;u,v} = f(s, X_s^{t,x;u,v}, \tilde{Y}_s^{t,x;u,v}, \tilde{Z}_s^{t,x;u,v}, \int_E \widetilde{K}_s^{t,x;u,v}(e) l(X_s^{t,x;u,v}, e) \lambda(de), u_s, v_s) ds \\
\qquad\quad - \tilde{Z}_s^{t,x;u,v} dB_s - \int_E \widetilde{K}_s^{t,x;u,v}(e) \tilde{\mu}(ds, de), \qquad s \in [t, t+\delta], \\
\tilde{Y}_{t+\delta}^{t,x;u,v} = \eta,
\end{cases}
\tag{3.15}
$$

where $X^{t,x;u,v}$ is the solution of SDE (3.1).

Remark 3.3. *When f is independent of (y, z, k) it holds that*

$$
G_{s,t+\delta}^{t,x;u,v}[\eta] = E[\eta + \int_s^{t+\delta} f(r, X_r^{t,x;u,v}, u_r, v_r) dr | \mathcal{F}_s], \quad s \in [t, t+\delta].
$$

Obviously, for the solution $(Y^{t,x;u,v}, Z^{t,x;u,v}, K^{t,x;u,v})$ of BSDE (3.5) we have

$$
G_{t,T}^{t,x;u,v}[\Phi(X_T^{t,x;u,v})] = G_{t,t+\delta}^{t,x;u,v}[Y_{t+\delta}^{t,x;u,v}].
\tag{3.16}
$$

Moreover,

$$
\begin{aligned}
J(t, x; u, v) &= Y_t^{t,x;u,v} = G_{t,T}^{t,x;u,v}[\Phi(X_T^{t,x;u,v})] = G_{t,t+\delta}^{t,x;u,v}[Y_{t+\delta}^{t,x;u,v}] \\
&= G_{t,t+\delta}^{t,x;u,v}[J(t+\delta, X_{t+\delta}^{t,x;u,v}; u, v)].
\end{aligned}
$$

Theorem 3.1. *Under the assumptions (H3.1) and (H3.2), the lower value function $W(t, x)$ obeys the following DPP : For any $0 \le t < t + \delta \le T$, $x \in \mathbb{R}^n$,*

$$
W(t, x) = \mathrm{essinf}_{\beta \in \mathcal{B}_{t,t+\delta}} \mathrm{esssup}_{u \in \mathcal{U}_{t,t+\delta}} G_{t,t+\delta}^{t,x;u,\beta(u)}[W(t+\delta, X_{t+\delta}^{t,x;u,\beta(u)})].
\tag{3.17}
$$

The proof is given in Section 6.2 of the Appendix since it is quite lengthy.

In Lemma 3.3 we have already seen that the lower value function $W(t, x)$ is Lipschitz continuous in x, uniformly in t. With the help of Theorem 3.1 we can now also study the continuity property of $W(t, x)$ in t.

Theorem 3.2. *Let us suppose that the assumptions (H3.1) and (H3.2) hold. Then the lower value function $W(t, x)$ is $\frac{1}{2}$−Hölder continuous in t: there exists a constant C such that, for every $x \in \mathbb{R}^n$, $t, t' \in [0, T]$,*

$$
|W(t, x) - W(t', x)| \le C(1 + |x|)|t - t'|^{\frac{1}{2}}.
$$

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Proof: Let $(t,x) \in [0,T] \times \mathbb{R}^n$ and $\delta > 0$ be arbitrarily given such that $0 < \delta \leq T - t$. Our objective is to prove the following inequality by using (6.21) and (6.22) in the Appendix:

$$- C(1 + |x|)\delta^{\frac{1}{2}} \leq W(t,x) - W(t+\delta, x) \leq C(1 + |x|)\delta^{\frac{1}{2}}. \tag{3.18}$$

From it we obtain immediately that W is $\frac{1}{2}$−Hölder continuous in t. We will only check the second inequality in (3.18), the first one can be shown in a similar way. To this end we note that due to (6.21), for an arbitrarily small $\varepsilon > 0$,

$$W(t,x) - W(t+\delta, x) \leq I_\delta^1 + I_\delta^2 + \varepsilon, \tag{3.19}$$

where

$$
\begin{aligned}
I_\delta^1 &:= G_{t,t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}[W(t+\delta, X_{t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)})] - G_{t,t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}[W(t+\delta,x)], \\
I_\delta^2 &:= G_{t,t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}[W(t+\delta,x)] - W(t+\delta, x),
\end{aligned}
$$

for arbitrarily chosen $\beta \in \mathcal{B}_{t,t+\delta}$ and $u^\varepsilon \in \mathcal{U}_{t,t+\delta}$ such that (6.21) holds. From Lemma 2.3 and the estimate (3.13)-(i) we obtain that, for some constant C independent of the controls u^ε and $\beta(u^\varepsilon)$,

$$
\begin{aligned}
|I_\delta^1| &\leq [CE(|W(t+\delta, X_{t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}) - W(t+\delta,x)|^2|\mathcal{F}_t)]^{\frac{1}{2}} \\
&\leq [CE(|X_{t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)} - x|^2|\mathcal{F}_t)]^{\frac{1}{2}},
\end{aligned}
$$

and since $E[|X_{t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)} - x|^2|\mathcal{F}_t] \leq C(1 + |x|^2)\delta$ we deduce that $|I_\delta^1| \leq C(1+|x|)\delta^{\frac{1}{2}}$. From the definition of $G_{t,t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}[\cdot]$ (see (3.14)) we know that the second term I_δ^2 can be written as

$$
\begin{aligned}
I_\delta^2 &= E[W(t+\delta,x) + \int_t^{t+\delta} f(s, X_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}, \widetilde{Y}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}, \widetilde{Z}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}, \int_E \widetilde{K}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}(e) \\
&\quad l(X_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}, e)\lambda(de), u_s^\varepsilon, \beta_s(u^\varepsilon))ds \\
&\quad - \int_t^{t+\delta} \widetilde{Z}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}dB_s - \int_t^{t+\delta} \int_E \widetilde{K}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}(e)\widetilde{\mu}(ds, de)|\mathcal{F}_t] - W(t+\delta,x) \\
&= E[\int_t^{t+\delta} f(s, X_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}, \widetilde{Y}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}, \widetilde{Z}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}, \int_E \widetilde{K}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}(e)l(X_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}, e)\lambda(de), \\
&\quad u_s^\varepsilon, \beta_s(u^\varepsilon))ds|\mathcal{F}_t],
\end{aligned}
$$

where $(\widetilde{Y}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}, \widetilde{Z}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}, \widetilde{K}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)})_{t \leq s \leq t+\delta}$ is the solution of BSDE (3.15) with the terminal condition $\eta = W(t+\delta, x)$. And with the help of the Schwartz inequality, the estimates (3.3) and (6.4)-(i) in the Appendix, we then have that, for some constant $C \in \mathbb{R}$ not depending on t and δ,

$$
\begin{aligned}
|I_\delta^2| &\leq \delta^{\frac{1}{2}} E[\int_t^{t+\delta} |f(s, X_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}, \widetilde{Y}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}, \widetilde{Z}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}, \int_E \widetilde{K}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}(e)l(X_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}, e) \\
&\quad \lambda(de), u_s^\varepsilon, \beta_s(u^\varepsilon))|^2 ds|\mathcal{F}_t]^{\frac{1}{2}} \\
&\leq \delta^{\frac{1}{2}} E[\int_t^{t+\delta} (|f(s, X_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}, 0, 0, 0, u_s^\varepsilon, \beta_s(u^\varepsilon))| + C|\widetilde{Y}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}| + C|\widetilde{Z}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}| \\
&\quad + C|\int_E \widetilde{K}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}(e)l(X_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}, e)\lambda(de)|)^2 ds|\mathcal{F}_t]^{\frac{1}{2}} \\
&\leq C\delta^{\frac{1}{2}} E[\int_t^{t+\delta} (1 + |X_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}|^2 + |\widetilde{Y}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}|^2 + |\widetilde{Z}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}|^2 \\
&\quad + \int_E |\widetilde{K}_s^{t,x;u^\varepsilon,\beta(u^\varepsilon)}(e)|^2 \lambda(de))ds|\mathcal{F}_t]^{\frac{1}{2}} \\
&\leq C(1 + |x|)\delta^{\frac{1}{2}}.
\end{aligned}
$$

Hence, from (3.19),

$$W(t,x) - W(t+\delta, x) \leq C(1 + |x|)\delta^{\frac{1}{2}} + \varepsilon,$$

and letting $\varepsilon \downarrow 0$ we get the second inequality of (3.18). The proof is complete. $\qquad\square$

4 Viscosity solutions of Isaacs' equations with integral-differential operators

In this section we consider the following second order integral-partial differential equations of Isaacs' type

$$
\begin{cases}
\frac{\partial}{\partial t}W(t,x) + H^-(t, x, W, DW, D^2W) = 0, & (t,x) \in [0,T) \times \mathbb{R}^n, \\
W(T, x) = \Phi(x), & x \in \mathbb{R}^n,
\end{cases} \tag{4.1}
$$

and

$$\begin{cases} \frac{\partial}{\partial t} U(t,x) + H^+(t,x,U,DU,D^2U) = 0, & (t,x) \in [0,T) \times \mathbb{R}^n, \\ U(T,x) = \Phi(x), & x \in \mathbb{R}^n. \end{cases} \tag{4.2}$$

Their Hamiltonians are given by

$$H^-(t,x,W,DW,D^2W) = \sup_{u \in U} \inf_{v \in V} H(t,x,W,DW,D^2W,u,v)$$

and

$$H^+(t,x,U,DU,D^2U) = \inf_{v \in V} \sup_{u \in U} H(t,x,U,DU,D^2U,u,v),$$

respectively, where

$$\begin{aligned} H(t,x,\Psi,D\Psi,D^2\Psi,u,v) = {} & \tfrac{1}{2} tr(\sigma\sigma^T(t,x,u,v)D^2\Psi) + D\Psi.b(t,x,u,v) \\ & + \int_E (\Psi(t,x+\gamma(t,x,u,v,e)) - \Psi(t,x) - D\Psi(t,x).\gamma(t,x,u,v,e))\lambda(de) \\ & + f(t,x,\Psi(t,x),D\Psi(t,x).\sigma(t,x,u,v), \int_E(\Psi(t,x+\gamma(t,x,u,v,e)) - \Psi(t,x))l(x,e)\lambda(de),u,v), \end{aligned}$$

$\Psi = W$ or U, resp., $(t,x,u,v) \in [0,T] \times \mathbb{R}^n \times U \times V$. Here the functions b, σ, f and Φ are supposed to satisfy (H3.1) and (H3.2), respectively.

In this section we want to prove that the lower value function $W(t,x)$ introduced by (3.9) is the viscosity solution of equation (4.1), while the upper value function $U(t,x)$ defined by (3.10) is the viscosity solution of equation (4.2). For this we translate Peng's BSDE approach [18] developed in the framework of stochastic control theory driven by Brownian motion into that of the stochastic differential games driven by Brownian motion and Poisson random measure. Uniqueness of the viscosity solution will be shown in the next section for the class of continuous functions satisfying some growth assumption which is weaker than the polynomial growth condition. We first recall the definition of a viscosity solution of equation (4.1). The definition is analogous for equation (4.2). The reader more interested in viscosity solutions is referred to Crandall, Ishii and Lions [6].

Remark 4.1. *We should assume here that there exists a constant C such that*

$$|\rho(e)| \le C(1 \wedge |e|), \text{ for all } e \in E. \tag{H4.1}$$

This assumption (H4.1) is only necessary for the definition 4.1 of the viscosity solution. If Definition 4.1 is restricted to W of linear growth then assumption (H4.1) is not necessary. Notice that W defined by (3.9) has this linear growth property.

Definition 4.1. *A real-valued continuous function $W \in C([0,T] \times \mathbb{R}^n)$ is called*
(i) *a viscosity subsolution of equation (4.1) if $W(T,x) \le \Phi(x)$, for all $x \in \mathbb{R}^n$, and if for all functions $\varphi \in C^3_{l,b}([0,T] \times \mathbb{R}^n)$ and $(t,x) \in [0,T) \times \mathbb{R}^n$ such that $W - \varphi$ attains a local maximum at (t,x),*

$$\begin{aligned} \frac{\partial \varphi}{\partial t}(t,x) + {} & sup_{u \in U} inf_{v \in V} \{A^{u,v}\varphi(t,x) + B^{\delta,u,v}(W,\varphi)(t,x) \\ & + f(t,x,W(t,x),D\varphi(t,x).\sigma(t,x,u,v),C^{\delta,u,v}(W,\varphi)(t,x),u,v)\} \ge 0, \end{aligned} \tag{4.3}$$

for any $\delta > 0$, where

$$A^{u,v}\varphi(t,x) = \frac{1}{2} tr(\sigma\sigma^T(t,x,u,v)D^2\varphi(t,x)) + D\varphi(t,x).b(t,x,u,v),$$

$$\begin{aligned} B^{\delta,u,v}(W,\varphi)(t,x) = {} & \int_{E_\delta}(\varphi(t,x+\gamma(t,x,u,v,e)) - \varphi(t,x) - D\varphi(t,x).\gamma(t,x,u,v,e))\lambda(de) \\ & + \int_{E_\delta^c}(W(t,x+\gamma(t,x,u,v,e)) - W(t,x) - D\varphi(t,x).\gamma(t,x,u,v,e))\lambda(de), \end{aligned}$$

and

$$\begin{aligned} C^{\delta,u,v}(W,\varphi)(t,x) = {} & \int_{E_\delta}(\varphi(t,x+\gamma(t,x,u,v,e)) - \varphi(t,x))l(x,e)\lambda(de) \\ & + \int_{E_\delta^c}(W(t,x+\gamma(t,x,u,v,e)) - W(t,x))l(x,e)\lambda(de), \end{aligned}$$

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with $E_\delta = \{e \in E | |e| < \delta\}$.

(ii) *a viscosity supersolution of equation (4.1) if $W(T, x) \geq \Phi(x)$, for all $x \in \mathbb{R}^n$, and if for all functions $\varphi \in C^3_{l,b}([0, T] \times \mathbb{R}^n)$ and $(t, x) \in [0, T) \times \mathbb{R}^n$ such that $W - \varphi$ attains a local minimum at (t, x),*

$$
\begin{aligned}
&\frac{\partial \varphi}{\partial t}(t, x) + \sup_{u \in U} \inf_{v \in V} \{ A^{u,v} \varphi(t, x) + B^{\delta, u, v}(W, \varphi)(t, x) \\
&\quad + f(t, x, W(t, x), D\varphi(t, x).\sigma(t, x, u, v), C^{\delta, u, v}(W, \varphi)(t, x), u, v) \} \leq 0.
\end{aligned}
\tag{4.4}
$$

(iii) *a viscosity solution of equation (4.1) if it is both a viscosity sub- and a supersolution of equation (4.1).*

Remark 4.2. $C^3_{l,b}([0, T] \times \mathbb{R}^n)$ *denotes the set of real-valued functions that are continuously differentiable up to the third order and whose derivatives of order from 1 to 3 are bounded.*

In analogy to [1] we have the following result:

Lemma 4.1. *In the definition of W being a viscosity sub- (resp., super-)solution of (4.1), we can replace*

$$
\begin{aligned}
B^{\delta, u, v}(W, \varphi)(t, x) &= B^{u,v} \varphi(t, x), \\
C^{\delta, u, v}(W, \varphi)(t, x) &= C^{u,v} \varphi(t, x),
\end{aligned}
$$

where

$$
\begin{aligned}
B^{u,v} \varphi(t, x) &= \int_E (\varphi(t, x + \gamma(t, x, u, v, e)) - \varphi(t, x) - D\varphi(t, x).\gamma(t, x, u, v, e))\lambda(de), \\
C^{u,v} \varphi(t, x) &= \int_E (\varphi(t, x + \gamma(t, x, u, v, e)) - \varphi(t, x))l(x, e)\lambda(de).
\end{aligned}
$$

Proof: We only consider the subsolution case, the supersolution case can be treated analogously.

If $(t, x) \in [0, T) \times \mathbb{R}^n$ such that $W - \varphi$ attains a global maximum at (t, x) we have $W(s, y) - \varphi(s, y) \leq W(t, x) - \varphi(t, x)$, for all $(s, y) \in [0, T] \times \mathbb{R}^n$. Therefore, $W(t, y) - W(t, x) \leq \varphi(t, y) - \varphi(t, x)$, for any $y \in \mathbb{R}^n$ and this yields, for any $\delta > 0$,

$$
\begin{aligned}
B^{\delta, u, v}(W, \varphi)(t, x) &\leq B^{u,v} \varphi(t, x), \\
C^{\delta, u, v}(W, \varphi)(t, x) &\leq C^{u,v} \varphi(t, x).
\end{aligned}
$$

Because f is increasing in k, from (4.3) we get

$$
\begin{aligned}
&\frac{\partial \varphi}{\partial t}(t, x) + \sup_{u \in U} \inf_{v \in V} \{ A^{u,v} \varphi(t, x) + B^{u,v} \varphi(t, x) \\
&\quad + f(t, x, W(t, x), D\varphi.\sigma(t, x, u, v), C^{u,v} \varphi(t, x), u, v) \} \geq 0.
\end{aligned}
\tag{4.5}
$$

It remains to show this last condition (4.5) implies (4.3). Changing φ into $\varphi - (\varphi(t, x) - W(t, x))$, we may assume that $W(t, x) = \varphi(t, x)$. Then $W(s, y) \leq \varphi(s, y)$, for all $(s, y) \in [0, T] \times \mathbb{R}^n$. Moreover, we may assume without loss of generality that,

 (i) for all $\alpha > 0$, there exists some $\eta_\alpha > 0$, with $\eta_\alpha \to 0$ as $\alpha \to 0$, such that, for all $(s, y) \in [0, T] \times \mathbb{R}^n$ with $|(s, y) - (t, x)| > \alpha$, $\varphi(s, y) - W(s, y) \geq \eta_\alpha$.

Furthermore, there exists a sequence of elements φ_α in $C^3_{l,b}([0, T] \times \mathbb{R}^n)$ with the following properties:

 (ii) $\varphi_\alpha(s, y) = \varphi(s, y) \geq W(s, y)$, if $|(s, y) - (t, x)| \notin (\alpha, \frac{1}{\alpha})$;

 (iii) $\varphi_\alpha(s, y) \geq W(s, y)$, if $\alpha \leq |(s, y) - (t, x)| \leq \frac{1}{\alpha}$;

 (iv) $\varphi_\alpha(s, y) \leq W(s, y) + \eta_\alpha$, if $3\alpha \leq |(s, y) - (t, x)| \leq \frac{1}{\alpha} - 2\alpha$;

 (v) $\varphi_\alpha(s, y) \leq \varphi(s, y)$, for all $(s, y) \in [0, T] \times \mathbb{R}^n$;

 (vi) There exists some $\bar{\rho}_\alpha > 0$ with $\bar{\rho}_\alpha \to 0$ ($\alpha \downarrow 0$) such that $0 \leq \varphi_\alpha(s, y) - W(s, y) \leq \bar{\rho}_\alpha$, for all (s, y) satisfying $|(s, y) - (t, x)| \leq \frac{1}{\alpha} - 2\alpha$.

Then, obviously, we have $D\varphi_\alpha(t, x) = D\varphi(t, x), \frac{\partial \varphi_\alpha(t,x)}{\partial t} = \frac{\partial \varphi(t,x)}{\partial t}, D^2\varphi_\alpha(t, x) = D^2\varphi(t, x)$. Thus, since $\varphi_\alpha(t, x) = W(t, x)$ and $\varphi_\alpha(s, y) \geq W(s, y)$, it follows from (4.5) that

$$
\begin{aligned}
&\frac{\partial \varphi_\alpha(t,x)}{\partial t} + \sup_{u \in U} \inf_{v \in V} \{ A^{u,v} \varphi_\alpha(t, x) + B^{u,v} \varphi_\alpha(t, x) \\
&\quad + f(t, x, W(t, x), D\varphi_\alpha.\sigma(t, x, u, v), C^{u,v} \varphi_\alpha(t, x), u, v) \} \geq 0.
\end{aligned}
$$

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From the above property (v) and the monotonicity of f, we get

$$\frac{\partial \varphi(t,x)}{\partial t} + \sup_{u \in U} \inf_{v \in V} \{A^{u,v}\varphi(t,x) + B^{\delta,u,v}(\varphi_\alpha, \varphi)(t,x)$$
$$+ f(t,x,W(t,x), D\varphi.\sigma(t,x,u,v), C^{\delta,u,v}(\varphi_\alpha, \varphi)(t,x), u, v)\} \geq 0.$$

Finally, by (vi) and the Lebesgue dominated convergence theorem we deduce that

$$\lim_{\alpha \to 0} \sup_{u \in U} \inf_{v \in V} B^{\delta,u,v}(\varphi_\alpha, \varphi)(t,x) = \sup_{u \in U} \inf_{v \in V} B^{\delta,u,v}(W, \varphi)(t,x);$$

$$\lim_{\alpha \to 0} \sup_{u \in U} \inf_{v \in V} C^{\delta,u,v}(\varphi_\alpha, \varphi)(t,x) = \sup_{u \in U} \inf_{v \in V} C^{\delta,u,v}(W, \varphi)(t,x).$$

Indeed, since $\varphi_\alpha(t,.)$, $W(t,.)$ are continuous and coincide in x we get

$$|B^{\delta,u,v}(\varphi_\alpha, \varphi)(t,x) - B^{\delta,u,v}(W, \varphi)(t,x)|$$
$$\leq \int_{E_\delta^c} |\varphi_\alpha(t, x + \gamma(t,x,u,v,e)) - W(t, x + \gamma(t,x,u,v,e))|\lambda(de)$$
$$\leq \bar{\rho}_\alpha \lambda(E_\delta^c) + C \int_{E_\delta^c} \mathbf{I}_{\{|\gamma(t,x,u,v,e)| > \frac{1}{\alpha} - 2\alpha\}} \lambda(de)$$
$$\longrightarrow 0 \ (\alpha \to 0) \text{ uniformly in } (u,v) \in U \times V.$$

The second convergence uses the same argument. Therefore, letting $\varepsilon \to 0$ in the above estimate yields the desired result. $\square$

Remark 4.3. *As concerns the construction of the sequence (φ_α), we also refer the reader to Remark 4.3 in Li and Peng [16].*

Let us now first prove that the lower value function $W(t,x)$ is a viscosity solution of equation (4.1).

Theorem 4.1. *Under the assumptions (H3.1) and (H3.2) the lower value function $W(t,x)$ is a viscosity solution of equation (4.1).*

For the proof of this theorem we need four auxiliary lemmas. To abbreviate notation we put, for some arbitrarily chosen but fixed $\varphi \in C_{l,b}^3([0,T] \times \mathbb{R}^n)$,

$$F(s,x,y,z,k,u,v) = \frac{\partial}{\partial s}\varphi(s,x) + A^{u,v}\varphi(s,x) + B^{u,v}\varphi(s,x)$$
$$+ f(s,x,y + \varphi(s,x), z + D\varphi(s,x).\sigma(s,x,u,v), \int_E k(e)l(x,e)\lambda(de) + C^{u,v}\varphi(s,x), u, v), \tag{4.6}$$

$(s,x,y,z,k,u,v) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times L^2(E, \mathcal{B}(E), \lambda; \mathbb{R}) \times U \times V$, and we consider the following BSDE defined on the interval $[t, t+\delta]$ $(0 < \delta \leq T - t)$:

$$\begin{cases} -dY_s^{1,u,v} = F(s, X_s^{t,x;u,v}, Y_s^{1,u,v}, Z_s^{1,u,v}, K_s^{1,u,v}, u_s, v_s)ds - Z_s^{1,u,v}dB_s - \int_E K_s^{1,u,v}(e)\widetilde{\mu}(ds,de), \\ Y_{t+\delta}^{1,u,v} = 0, \end{cases} \tag{4.7}$$

where the process $X^{t,x,u,v}$ has been introduced by equation (3.1) and $u(\cdot) \in \mathcal{U}_{t,t+\delta}$, $v(\cdot) \in \mathcal{V}_{t,t+\delta}$.

Remark 4.4. *It is not hard to check that $F(s, X_s^{t,x;u,v}, y, z, k, u_s, v_s)$ satisfies (A1) and (A2). Thus, due to Lemma 2.1, equation (4.7) has a unique solution.*

We can characterize the solution process $Y^{1,u,v}$ as follows:

Lemma 4.2. *For every $s \in [t, t+\delta]$, we have the following relationship:*

$$Y_s^{1,u,v} = G_{s,t+\delta}^{t,x;u,v}[\varphi(t+\delta, X_{t+\delta}^{t,x;u,v})] - \varphi(s, X_s^{t,x;u,v}), \qquad P\text{-a.s.} \tag{4.8}$$

Proof: We recall that $G_{s,t+\delta}^{t,x;u,v}[\varphi(t+\delta, X_{t+\delta}^{t,x;u,v})]$ is defined with the help of the solution of the BSDE

$$\begin{cases} -dY_s^{u,v} = f(s, X_s^{t,x;u,v}, Y_s^{u,v}, Z_s^{u,v}, \int_E K_s^{u,v}(e)l(X_s^{t,x;u,v}, e)\lambda(de), u_s, v_s)ds - Z_s^{u,v}dB_s \\ \qquad - \int_E K_s^{u,v}(e)\widetilde{\mu}(ds,de), \qquad\qquad\qquad s \in [t, t+\delta], \\ Y_{t+\delta}^{u,v} = \varphi(t+\delta, X_{t+\delta}^{t,x;u,v}), \end{cases}$$

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by the following formula:

$$G_{s,t+\delta}^{t,x;u,v}[\varphi(t+\delta, X_{t+\delta}^{t,x;u,v})] = Y_s^{u,v}, \quad s \in [t, t+\delta] \tag{4.9}$$

(see (3.14)). Therefore, we only need to prove that $Y_s^{u,v} - \varphi(s, X_s^{t,x;u,v}) \equiv Y_s^{1,u,v}$. This result can be obtained easily by applying Itô's formula to $\varphi(s, X_s^{t,x;u,v})$. Indeed, we get that the stochastic differentials of $Y_s^{u,v} - \varphi(s, X_s^{t,x;u,v})$ and $Y_s^{1,u,v}$ coincide, while at the terminal time $t+\delta$, $Y_{t+\delta}^{u,v} - \varphi(t+\delta, X_{t+\delta}^{t,x;u,v}) = 0 = Y_{t+\delta}^{1,u,v}$. So the proof is complete. □

Now we consider the following simple BSDE in which the driving process $X^{t,x;u,v}$ is replaced by its deterministic initial value x:

$$\begin{cases} -dY_s^{2,u,v} = F(s, x, Y_s^{2,u,v}, Z_s^{2,u,v}, K_s^{2,u,v}, u_s, v_s)ds - Z_s^{2,u,v}dB_s - \int_E K_s^{2,u,v}(e)\widetilde{\mu}(ds, de), \\ Y_{t+\delta}^{2,u,v} = 0, \qquad\qquad\qquad s \in [t, t+\delta], \end{cases} \tag{4.10}$$

where $u(\cdot) \in \mathcal{U}_{t,t+\delta}$, $v(\cdot) \in \mathcal{V}_{t,t+\delta}$. The following lemma will allow us to neglect the difference $|Y_t^{1,u,v} - Y_t^{2,u,v}|$ for sufficiently small $\delta > 0$.

Lemma 4.3. *For every $u \in \mathcal{U}_{t,t+\delta}$, $v \in \mathcal{V}_{t,t+\delta}$, we have*

$$|Y_t^{1,u,v} - Y_t^{2,u,v}| \le C\delta^{\frac{3}{2}}, \quad P\text{-a.s.}, \tag{4.11}$$

where C is independent of the control processes u and v.

Proof: From Proposition 1.1 in [1], we have for all $p \ge 2$ the existence of some $C_p \in \mathbb{R}^+$ such that

$$E[\sup_{t \le s \le t+\delta} |X_s^{t,x;u,v} - x|^p |\mathcal{F}_t] \le C_p \delta(1 + |x|^p), \quad P\text{-a.s.,} \quad \text{uniformly in } u \in \mathcal{U}_{t,t+\delta}, v \in \mathcal{V}_{t,t+\delta}. \tag{4.12}$$

We now apply Lemma 2.3 combined with (4.12) to equations (4.7) and (4.10). For this we set in Lemma 2.3:

$$\xi_1 = \xi_2 = 0, \ g(s, y, z) = F(s, X_s^{t,x,u,v}, y, z, k, u_s, v_s),$$

$$\varphi_1(s) = 0, \ \varphi_2(s) = F(s, x, Y_s^{2,u,v}, Z_s^{2,u,v}, K_s^{2,u,v}, u_s, v_s) - F(s, X_s^{t,x,u,v}, Y_s^{2,u,v}, Z_s^{2,u,v}, K_s^{2,u,v}, u_s, v_s).$$

Obviously, the function g is Lipschitz with respect to (y, z, k). We also notice that

$$\begin{aligned} B^{u,v}\varphi(s, x) &= \int_E (\varphi(s, x + \gamma(s, x, u, v, e)) - \varphi(s, x) - D\varphi \cdot \gamma(s, x, u, v, e))\lambda(de) \\ &= \int_E \int_0^1 (1-\theta)tr(D^2\varphi(s, x + \theta\gamma(s, x, u, v, e))\gamma\gamma^T(s, x, u, v, e))d\theta\lambda(de); \\ C^{u,v}\varphi(s, x) &= \int_E (\varphi(s, x + \gamma(s, x, u, v, e)) - \varphi(s, x))l(x, e)\lambda(de) \\ &= \int_E \int_0^1 D\varphi(s, x + \theta\gamma(s, x, u, v, e))\gamma(s, x, u, v, e)d\theta l(x, e)\lambda(de). \end{aligned}$$

Then, we can get $|\varphi_2(s)| \le C(1 + |x|^2)(|X_s^{t,x;u,v} - x| + |X_s^{t,x;u,v} - x|^3)$, for $s \in [t, t+\delta]$, $(t, x) \in [0, T] \times \mathbb{R}^n$, $u \in \mathcal{U}_{t,t+\delta}, v \in \mathcal{V}_{t,t+\delta}$. Thus, with the notation $\rho_0(r) = (1 + |x|^2)(r + r^3)$, $r \ge 0$, we have

$$\begin{aligned} &E[\int_t^{t+\delta}(|Y_s^{1,u,v} - Y_s^{2,u,v}|^2 + |Z_s^{1,u,v} - Z_s^{2,u,v}|^2)ds|\mathcal{F}_t] \\ &\qquad + E[\int_t^{t+\delta}\int_E |K_s^{1,u,v}(e) - K_s^{2,u,v}(e)|^2\lambda(de)ds|\mathcal{F}_t] \\ &\le CE[\int_t^{t+\delta}\rho_0^2(|X_s^{t,x,u,v} - x|)ds|\mathcal{F}_t] \\ &\le C\delta E[\sup_{t \le s \le t+\delta}\rho_0^2(|X_s^{t,x,u,v} - x|)|\mathcal{F}_t] \\ &\le C\delta^2. \end{aligned}$$

Therefore,

$$\begin{aligned} &|Y_t^{1,u,v} - Y_t^{2,u,v}| = |E[(Y_t^{1,u,v} - Y_t^{2,u,v})|\mathcal{F}_t]| \\ &= |E[\int_t^{t+\delta}(F(s, X_s^{t,x,u,v}, Y_s^{1,u,v}, Z_s^{1,u,v}, K_s^{1,u,v}, u_s, v_s) \\ &\qquad\qquad -F(s, x, Y_s^{2,u,v}, Z_s^{2,u,v}, K_s^{2,u,v}, u_s, v_s))ds|\mathcal{F}_t]| \\ &\le CE[\int_t^{t+\delta}(\rho_0(|X_s^{t,x,u,v} - x|) + |Y_s^{1,u,v} - Y_s^{2,u,v}| + |Z_s^{1,u,v} - Z_s^{2,u,v}|)ds|\mathcal{F}_t] \\ &\qquad + CE[\int_t^{t+\delta}|\int_E(K_s^{1,u,v}(e) - K_s^{2,u,v}(e))l(x, e)\lambda(de)|ds|\mathcal{F}_t] \\ &\le CE[\int_t^{t+\delta}\rho_0(|X_s^{t,x,u,v} - x|)ds|\mathcal{F}_t] + C\delta^{\frac{1}{2}}\{E[\int_t^{t+\delta}|Y_s^{1,u,v} - Y_s^{2,u,v}|^2ds|\mathcal{F}_t]^{\frac{1}{2}} \\ &\qquad + E[\int_t^{t+\delta}|Z_s^{1,u,v} - Z_s^{2,u,v}|^2ds|\mathcal{F}_t]^{\frac{1}{2}} + E[\int_t^{t+\delta}\int_E |K_s^{1,u,v}(e) - K_s^{2,u,v}(e)|^2\lambda(de)ds|\mathcal{F}_t]^{\frac{1}{2}}\} \\ &\le C\delta^{\frac{3}{2}}. \end{aligned}$$

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Thus, the proof is complete. □

Lemma 4.4. *Let $Y_0(\cdot)$ be the solution of the following ordinary differential equation:*

$$\begin{cases} -\dot{Y}_0(s) & = & F_0(s, x, Y_0(s), 0, 0), \quad s \in [t, t+\delta], \\ Y_0(t+\delta) & = & 0, \end{cases} \tag{4.13}$$

where the function F_0 is defined by

$$F_0(s, x, y, z, k) = sup_{u \in U} inf_{v \in V} F(s, x, y, z, k, u, v), \\ (s, x, y, z, k) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times L^2(E, \mathcal{B}(E), \lambda; \mathbb{R}). \tag{4.14}$$

Then, P-a.s.,

$$esssup_{u \in \mathcal{U}_{t,t+\delta}} essinf_{v \in \mathcal{V}_{t,t+\delta}} Y_t^{2,u,v} = Y_0(t). \tag{4.15}$$

Proof: Obviously, $F_0(s, x, y, z, k)$ is Lipschitz in (y, z, k), uniformly with respect to (s, x). This guarantees the existence and uniqueness for equation (4.13). We first introduce the function

$$F_1(s, x, y, z, k, u) = \inf_{v \in V} F(s, x, y, z, k, u, v), \\ (s, x, y, z, k, u) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times L^2(E, \mathcal{B}(E), \lambda; \mathbb{R}) \times U, \tag{4.16}$$

and consider the BSDE

$$\begin{cases} -dY_s^{3,u} & = & F_1(s, x, Y_s^{3,u}, Z_s^{3,u}, K_s^{3,u}, u_s)ds - Z_s^{3,u}dB_s - \int_E K_s^{3,u}(e)\widetilde{\mu}(ds, de), \\ Y_{t+\delta}^{3,u} & = & 0, \quad s \in [t, t+\delta], \end{cases} \tag{4.17}$$

for $u \in \mathcal{U}_{t,t+\delta}$. We notice that since $F_1(s, x, y, z, k, u_s)$ is Lipschitz in (y, z, k), for every $u \in \mathcal{U}_{t,t+\delta}$, there exists a unique solution $(Y^{3,u}, Z^{3,u}, K^{3,u})$ to the BSDE (4.17). Moreover,

$$Y_t^{3,u} = essinf_{v(\cdot) \in \mathcal{V}_{t,t+\delta}} Y_t^{2,u,v}, \text{ P-a.s., for all } u \in \mathcal{U}_{t,t+\delta}.$$

Indeed, from the definition of F_1 and Lemma 2.2 (comparison theorem) we have

$$Y_t^{3,u} \le essinf_{v(\cdot) \in \mathcal{V}_{t,t+\delta}} Y_t^{2,u,v}, \text{ P-a.s., for all } u \in \mathcal{U}_{t,t+\delta}.$$

On the other hand, there exists a measurable function $v^3 : [t, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times U \to V$ such that

$$F_1(s, x, y, z, k, u) = F(s, x, y, z, k, u, v^3(s, x, y, z, k, u)), \text{ for any } s, x, y, z, k, u.$$

Then, given an arbitrary $u \in \mathcal{U}_{t,t+\delta}$ we put

$$\widetilde{v}_s^3 := v^3(s, x, Y_s^{3,u}, Z_s^{3,u}, K_s^{3,u}, u_s), \ s \in [t, t+\delta],$$

and observe that $\widetilde{v}^3 \in \mathcal{V}_{t,t+\delta}$, and

$$F_1(s, x, Y_s^{3,u}, Z_s^{3,u}, K_s^{3,u}, u_s) = F(s, x, Y_s^{3,u}, Z_s^{3,u}, K_s^{3,u}, u_s, \widetilde{v}_s^3), \ s \in [t, t+\delta].$$

Consequently, from the uniqueness of the solution of the BSDE it follows that

$$(Y^{3,u}, Z^{3,u}, K^{3,u}) = (Y^{2,u,\widetilde{v}^3}, Z^{2,u,\widetilde{v}^3}, K^{2,u,\widetilde{v}^3})$$

and, in particular, $Y_t^{3,u} = Y_t^{2,u,\widetilde{v}^3}$, P-a.s. This proves that

$$Y_t^{3,u} = essinf_{v \in \mathcal{V}_{t,t+\delta}} Y_t^{2,u,v}, \text{ P-a.s., for all } u \in \mathcal{U}_{t,t+\delta}.$$

Finally, since $F_0(s, x, y, z, k) = sup_{u \in U} F_1(s, x, y, z, k, u),$ an argument similar to that developed above yields

$$Y_0(t) = esssup_{u \in \mathcal{U}_{t,t+\delta}} Y_t^{3,u} (= esssup_{u \in \mathcal{U}_{t,t+\delta}} essinf_{v \in \mathcal{V}_{t,t+\delta}} Y_t^{2,u,v}), \text{ P-a.s.}$$

It uses the fact that equation (4.13) can be considered as a BSDE with solution $(Y_s, Z_s, K_s) = (Y_0(s), 0, 0)$. The proof is complete. □

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Lemma 4.5. *For every $u \in \mathcal{U}_{t,t+\delta}, v \in \mathcal{V}_{t,t+\delta}$, we have*

$$E[\int_t^{t+\delta} |Y_s^{2,u,v}| ds | \mathcal{F}_t] + E[\int_t^{t+\delta} |Z_s^{2,u,v}| ds | \mathcal{F}_t] + E[\int_t^{t+\delta} | \int_E K_s^{2,u,v}(e) l(x,e) \lambda(de)| ds | \mathcal{F}_t]$$
$$\leq C\delta^{\frac{3}{2}}, \ \ P\text{-}a.s., \tag{4.18}$$

where the constant C is independent of t, δ and the control processes u, v.

Proof: Since $F(s,x,\cdot,\cdot,\cdot,u,v)$ has a linear growth in (y,z,k), uniformly in (u,v), we get from Lemma 2.3 that, for some constant C independent of δ and the control processes u, v, P-a.s.,

$$|Y_s^{2,u,v}|^2 \leq C\delta, \ \ E[\int_s^{t+\delta} |Z_r^{2,u,v}|^2 dr | \mathcal{F}_s] \leq C\delta,$$

$$E[\int_s^{t+\delta} \int_E |K_r^{2,u,v}(e)|^2 \lambda(de) dr | \mathcal{F}_s] \leq C\delta, \ t \leq s \leq t+\delta.$$

On the other hand, from equation (4.10),

$$
\begin{aligned}
|Y_s^{2,u,v}| &\leq E[\int_s^{t+\delta} |F(r,x,Y_r^{2,u,v}, Z_r^{2,u,v}, K_r^{2,u,v}, u_r, v_r)| dr | \mathcal{F}_s] \\
&\leq CE[\int_s^{t+\delta} (1 + |x|^2 + |Y_r^{2,u,v}| + |Z_r^{2,u,v}| + | \int_E K_r^{2,u,v}(e) l(x,e) \lambda(de)|) dr | \mathcal{F}_s] \\
&\leq C\delta + C\sqrt{\delta}(E[\int_s^{t+\delta} |Z_r^{2,u,v}|^2 dr | \mathcal{F}_s])^{\frac{1}{2}} + C\sqrt{\delta}(E[\int_s^{t+\delta} \int_E |K_r^{2,u,v}(e)|^2 \lambda(de) dr | \mathcal{F}_s])^{\frac{1}{2}} \\
&\leq C\delta, \ P\text{-}a.s., \ \ s \in [t, t+\delta],
\end{aligned}
$$

and, applying Itô formula to $|Y_s^{2,u,v}|^2$ we can get

$$E[\int_t^{t+\delta} |Z_s^{2,u,v}|^2 ds | \mathcal{F}_t] + E[\int_t^{t+\delta} \int_E |K_s^{2,u,v}(e)|^2 \lambda(de) ds | \mathcal{F}_t] \leq C\delta^2, \ \ P\text{-}a.s.$$

Finally,

$$
\begin{aligned}
&E[\int_t^{t+\delta} |Y_s^{2,u,v}| ds | \mathcal{F}_t] + E[\int_t^{t+\delta} |Z_s^{2,u,v}| ds | \mathcal{F}_t] + E[\int_t^{t+\delta} | \int_E K_s^{2,u,v}(e) l(x,e) \lambda(de)| ds | \mathcal{F}_t] \\
&\leq C\delta^2 + \delta^{\frac{1}{2}} \{E[\int_t^{t+\delta} |Z_s^{2,u,v}|^2 ds | \mathcal{F}_t]\}^{\frac{1}{2}} + C\delta^{\frac{1}{2}} \{E[\int_t^{t+\delta} \int_E |K_s^{2,u,v}(e)|^2 \lambda(de) ds | \mathcal{F}_t]\}^{\frac{1}{2}} \\
&\leq C\delta^{\frac{3}{2}}, \ \ P\text{-}a.s.
\end{aligned}
$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Now we are able to give the proof of Theorem 4.1:

Proof: (1) Obviously, $W(T,x) = \Phi(x)$, $x \in \mathbb{R}^n$. Let us show in a first step that W is a viscosity supersolution. For this we suppose that $\varphi \in C_{l,b}^3([0,T] \times \mathbb{R}^n)$, and $(t,x) \in [0,T) \times \mathbb{R}^n$ are such that $W - \varphi$ attains a minimum at (t,x). Notice that we can replace the condition of a local minimum by that of a global one in the definition of the viscosity supersolution since W is continuous and of at most linear growth. Without loss of generality we may also suppose that $\varphi(t,x) = W(t,x)$. Then, due to the DPP (see Theorem 3.1),

$$\varphi(t,x) = W(t,x) = \operatorname{essinf}_{\beta \in \mathcal{B}_{t,t+\delta}} \operatorname{esssup}_{u \in \mathcal{U}_{t,t+\delta}} G_{t,t+\delta}^{t,x;u,\beta(u)}[W(t+\delta, X_{t+\delta}^{t,x;u,\beta(u)})], \ 0 \leq \delta \leq T - t,$$

and from $W \geq \varphi$ and the monotonicity property of $G_{t,t+\delta}^{t,x;u,\beta(u)}[\cdot]$ (see Lemma 2.2) we obtain

$$\operatorname{essinf}_{\beta \in \mathcal{B}_{t,t+\delta}} \operatorname{esssup}_{u \in \mathcal{U}_{t,t+\delta}} \{G_{t,t+\delta}^{t,x;u,\beta(u)}[\varphi(t+\delta, X_{t+\delta}^{t,x;u,\beta(u)})] - \varphi(t,x)\} \leq 0, \ P\text{-}a.s.$$

Thus, from Lemma 4.2,

$$\operatorname{essinf}_{\beta \in \mathcal{B}_{t,t+\delta}} \operatorname{esssup}_{u \in \mathcal{U}_{t,t+\delta}} Y_t^{1,u,\beta(u)} \leq 0, \ P\text{-}a.s.,$$

and further, from Lemma 4.3 we have

$$\operatorname{essinf}_{\beta \in \mathcal{B}_{t,t+\delta}} \operatorname{esssup}_{u \in \mathcal{U}_{t,t+\delta}} Y_t^{2,u,\beta(u)} \leq C\delta^{\frac{3}{2}}, \ P\text{-}a.s.$$

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Consequently, since $\mathrm{essinf}_{v\in\mathcal{V}_{t,t+\delta}}Y_t^{2,u,v}\leq Y_t^{2,u,\beta(u)}$, $\beta\in\mathcal{B}_{t,t+\delta}$, we get

$$\mathrm{esssup}_{u\in\mathcal{U}_{t,t+\delta}}\mathrm{essinf}_{v\in\mathcal{V}_{t,t+\delta}}Y_t^{2,u,v}\leq\mathrm{essinf}_{\beta\in\mathcal{B}_{t,t+\delta}}\mathrm{esssup}_{u\in\mathcal{U}_{t,t+\delta}}Y_t^{2,u,\beta(u)}\leq C\delta^{\frac{3}{2}},\ \text{P-a.s.},$$

and Lemma 4.4 implies

$$Y_0(t)\leq C\delta^{\frac{3}{2}},\ \text{P-a.s.},$$

where Y_0 is the unique solution of equation (4.13). It then follows easily that

$$\sup_{u\in U}\inf_{v\in V}F(t,x,0,0,0,u,v)=F_0(t,x,0,0,0)\leq 0,$$

and from the definition of F we see that W is a viscosity supersolution of equation (4.1).

(2) The second step is devoted to the proof that W is a viscosity subsolution. For this we suppose that $\varphi\in C_{l,b}^3([0,T]\times\mathbb{R}^n)$ and $(t,x)\in[0,T)\times\mathbb{R}^n$ are such that $W-\varphi$ attains a maximum at (t,x). Without loss of generality we suppose again $\varphi(t,x)=W(t,x)$. We must prove that

$$\sup_{u\in U}\inf_{v\in V}F(t,x,0,0,0,u,v)=F_0(t,x,0,0,0)\geq 0.$$

Let us suppose that this is not true. Then there exists some $\theta>0$ such that

$$F_0(t,x,0,0,0)=\sup_{u\in U}\inf_{v\in V}F(t,x,0,0,0,u,v)\leq-\theta<0,\tag{4.19}$$

and we can find a measurable function $\psi:U\to V$ such that

$$F(t,x,0,0,0,u,\psi(u))\leq-\frac{3}{4}\theta,\ \text{for all }u\in U.$$

Moreover, since $F(\cdot,x,0,0,0,\cdot,\cdot)$ is uniformly continuous on $[0,T]\times U\times V$ there exists some $T-t\geq R>0$ such that

$$F(s,x,0,0,0,u,\psi(u))\leq-\frac{1}{2}\theta,\ \text{for all }u\in U\text{and }|s-t|\leq R.\tag{4.20}$$

On the other hand, due to the DPP (see Theorem 3.1), for every $\delta\in(0,R]$,

$$\varphi(t,x)=W(t,x)=\mathrm{essinf}_{\beta\in\mathcal{B}_{t,t+\delta}}\mathrm{esssup}_{u\in\mathcal{U}_{t,t+\delta}}G_{t,t+\delta}^{t,x;u,\beta(u)}[W(t+\delta,X_{t+\delta}^{t,x;u,\beta(u)})],$$

and from $W\leq\varphi$ and the monotonicity property of $G_{t,t+\delta}^{t,x;u,\beta(u)}[\cdot]$ (see Lemma 2.2) we obtain

$$\mathrm{essinf}_{\beta\in\mathcal{B}_{t,t+\delta}}\mathrm{esssup}_{u\in\mathcal{U}_{t,t+\delta}}\{G_{t,t+\delta}^{t,x;u,\beta(u)}[\varphi(t+\delta,X_{t+\delta}^{t,x;u,\beta(u)})]-\varphi(t,x)\}\geq 0,\ \text{P-a.s.}$$

Thus, from Lemma 4.2,

$$\mathrm{essinf}_{\beta\in\mathcal{B}_{t,t+\delta}}\mathrm{esssup}_{u\in\mathcal{U}_{t,t+\delta}}Y_t^{1,u,\beta(u)}\geq 0,\ \text{P-a.s.},$$

and, in particular,

$$\mathrm{esssup}_{u\in\mathcal{U}_{t,t+\delta}}Y_t^{1,u,\psi(u)}\geq 0,\ \text{P-a.s.}$$

Here, by putting $\psi_s(u)(\omega)=\psi(u_s(\omega))$, $(s,\omega)\in[t,T]\times\Omega$, we identify ψ as an element of $\mathcal{B}_{t,t+\delta}$. Given an arbitrarily $\varepsilon>0$ we can choose $u^\varepsilon\in\mathcal{U}_{t,t+\delta}$ such that $Y_t^{1,u^\varepsilon,\psi(u^\varepsilon)}\geq-\varepsilon\delta$ (similar to the proof of (6.21)). From Lemma 4.3 we further have

$$Y_t^{2,u^\varepsilon,\psi(u^\varepsilon)}\geq-C\delta^{\frac{3}{2}}-\varepsilon\delta,\ \text{P-a.s.}\tag{4.21}$$

Taking into account that

$$Y_t^{2,u^\varepsilon,\psi(u^\varepsilon)}=E[\int_t^{t+\delta}F(s,x,Y_s^{2,u^\varepsilon,\psi(u^\varepsilon)},Z_s^{2,u^\varepsilon,\psi(u^\varepsilon)},K_s^{2,u^\varepsilon,\psi(u^\varepsilon)},u_s^\varepsilon,\psi_s(u_\cdot^\varepsilon))ds|\mathcal{F}_t]$$

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we get from the Lipschitz property of F in (y, z, k), (4.20) and Lemma 4.5 that

$$
\begin{aligned}
Y_t^{2,u^\varepsilon,\psi(u^\varepsilon)} &\leq E[\int_t^{t+\delta}(C|Y_s^{2,u^\varepsilon,\psi(u^\varepsilon)}| + C|Z_s^{2,u^\varepsilon,\psi(u^\varepsilon)}| + C|\int_E K_s^{2,u^\varepsilon,\psi(u^\varepsilon)}(e)l(x,e)\lambda(de)| \\
&\qquad + F(s,x,0,0,0,u_s^\varepsilon,\psi_s(u_{\cdot}^\varepsilon)))ds|\mathcal{F}_t] \\
&\leq C\delta^{\frac{3}{2}} - \tfrac{1}{2}\theta\delta, \text{ P-a.s.}
\end{aligned}
\tag{4.22}
$$

From (4.21) and (4.22), $-C\delta^{\frac{1}{2}} - \varepsilon \leq C\delta^{\frac{1}{2}} - \frac{1}{2}\theta$, P-a.s. Letting $\delta \downarrow 0$, and then $\varepsilon \downarrow 0$ we deduce $\theta \leq 0$ which induces a contradiction. Therefore,

$$
F_0(t,x,0,0,0) = \sup_{u \in U}\inf_{v \in V} F(t,x,0,0,0,u,v) \geq 0,
$$

and from the definition of F, we know that W is a viscosity subsolution of equation (4.1). Finally, the results from the first and the second step prove that W is a viscosity solution of equation (4.1). $\qquad\square$

Remark 4.5. *Similarly, we can prove that U is a viscosity solution of equation (4.2).*

5 Viscosity Solution of Isaacs' Equation: Uniqueness Theorem

The objective of this section is to study the uniqueness of the viscosity solution of Isaacs' equation (4.1),

$$
\begin{cases}
\frac{\partial}{\partial t}W(t,x) + H^-(t,x,W,DW,D^2W) = 0, & (t,x) \in [0,T) \times \mathbb{R}^n, \\
W(T,x) = \Phi(x), & x \in \mathbb{R}^n.
\end{cases}
\tag{5.1}
$$

Recall that

$$
\begin{aligned}
H^-(t,x,W,DW,D^2W) &= \sup_{u \in U}\inf_{v \in V}\{\tfrac{1}{2}tr(\sigma\sigma^T(t,x,u,v)D^2W) + DW.b(t,x,u,v) \\
&\quad + \int_E (W(t,x+\gamma(t,x,u,v,e)) - W(t,x) - DW.\gamma(t,x,u,v,e))\lambda(de) \\
&\quad + f(t,x,W(t,x),DW.\sigma(t,x,u,v),\int_E(W(t,x+\gamma(t,x,u,v,e)) - W(t,x))l(x,e)\lambda(de),u,v)\},
\end{aligned}
$$

where $t \in [0,T]$, $x \in \mathbb{R}^n$. The functions b, σ, f and Φ are still supposed to satisfy (H3.1) and (H3.2), respectively.

We will prove the uniqueness for equation (5.1) in the following space of continuous functions
$\Theta = \{\varphi \in C([0,T] \times \mathbb{R}^n) : \exists \widetilde{A} > 0 \text{ such that}$

$$
\lim_{|x| \to \infty} \varphi(t,x)\exp\{-\widetilde{A}[\log((|x|^2 + 1)^{\frac{1}{2}})]^2\} = 0, \text{ uniformly in } t \in [0,T]\}.
$$

This space of continuous functions is endowed with a growth condition which is slightly weaker than the assumption of polynomial growth but more restrictive than that of exponential growth. This growth condition was introduced by Barles, Buckdahn and Pardoux [1] and Barles and Imbert [2] to prove the uniqueness of the viscosity solution of an integral-partial differential equation associated with a decoupled FBSDE with jumps but without controls. It was shown in [1] that this kind of growth condition is optimal for the uniqueness and can not be weakened in general. We adapt the ideas developed in [1] to Isaacs' equation (5.1) to prove the uniqueness of the viscosity solution in Θ. Since the proof of the uniqueness in Θ for equation (4.2) is essentially the same we will restrict ourselves to that of (5.1). Before stating the main result of this section, let us begin with two auxiliary lemmata. Denoting by K a Lipschitz constant of $f(t,.,.,.,.,u,v)$, which is uniformly in (t,u,v), we have the following

Lemma 5.1. *Let $u_1 \in \Theta$ be a viscosity subsolution and $u_2 \in \Theta$ be a viscosity supersolution of equation (5.1). Then the function $\omega := u_1 - u_2$ is a viscosity subsolution of the equation*

$$
\begin{cases}
\frac{\partial}{\partial t}\omega(t,x) + sup_{u \in U, v \in V}\{A^{u,v}\omega(t,x) + B^{u,v}\omega(t,x) + K|\omega(t,x)| + K|D\omega(t,x).\sigma(t,x,u,v)| \\
\qquad + K(C^{u,v}\omega(t,x))^+\} = 0, & (t,x) \in [0,T) \times \mathbb{R}^n, \\
\omega(T,x) = 0, & x \in \mathbb{R}^n.
\end{cases}
\tag{5.2}
$$

The proof of this lemma follows directly from that of Lemma 3.7 in [1] with the help of Lemma 1 (Nonlocal Jensen-Ishii's Lemma) in Barles and Imbert [2].

Now we can prove the uniqueness theorem.

Theorem 5.1. *We assume that (H3.1) and (H3.2) hold. Let u_1 (resp., u_2) $\in \Theta$ be a viscosity subsolution (resp., supersolution) of equation (5.1). Then we have*

$$u_1(t,x) \leq u_2(t,x), \quad for\ all\ \ (t,x) \in [0,T] \times \mathbb{R}^n. \tag{5.3}$$

Proof. Let us first suppose that u_1 and u_2 are bounded and put $\omega_1 := u_1 - u_2$. Theorem 4.1 in [2] establishes a comparison principle for bounded sub- and supersolutions of Hamilton-Jacobi-Bellman equations with nonlocal term of type (5.2). We know from Lemma 5.1 that ω_1 is a viscosity subsolution of equation (5.2). On the other hand, $\omega_2 = 0$ is, obviously, a viscosity solution and, hence, also a viscosity supersolution of equation (5.2). Both functions ω_1 and ω_2 are bounded, and the comparison principle stated in Theorem 4.1 in [2] yields that $u_1 - u_2 = \omega_1 \leq \omega_2 = 0$, i.e., $u_1 \leq u_2$ on $[0,T] \times R^n$. Finally, if u_1, u_2 are viscosity solutions of (5.2), they are both viscosity sub- and supersolution, and from the just proved comparison result we get the equality of u_1 and u_2. However, under our stand assumptions we can not expect that W is bounded, so that we have to prove the theorem for $u_1, u_2 \in \Theta$. For the proof the following auxiliary lemma is needed.

In analogy to [1] we also have

Lemma 5.2. *For any $\widetilde{A} > 0$, there exists $C_1 > 0$ such that the function*

$$\chi(t,x) = \exp[(C_1(T-t) + \widetilde{A})\psi(x)],$$

with

$$\psi(x) = [\log((|x|^2 + 1)^{\frac{1}{2}}) + 1]^2, \ x \in \mathbb{R}^n,$$

satisfies

$$\frac{\partial}{\partial t}\chi(t,x) + \sup_{u \in U, v \in V}\{A^{u,v}\chi(t,x) + B^{u,v}\chi(t,x) + K\chi(t,x)+ \\ K|D\chi(t,x).\sigma(t,x,u,v)| + K(C^{u,v}\chi(t,x))^+\} < 0 \ \ in\ [t_1,T] \times \mathbb{R}^n,\ where\ \ t_1 = T - \frac{\widetilde{A}}{C_1}. \tag{5.4}$$

Proof. By direct calculus we first deduce the following estimates for the first and second derivatives of ψ:

$$|D\psi(x)| \leq \frac{2[\psi(x)]^{\frac{1}{2}}}{(|x|^2 + 1)^{\frac{1}{2}}} \leq 4, \quad |D^2\psi(x)| \leq \frac{C(1 + [\psi(x)]^{\frac{1}{2}})}{|x|^2 + 1}, \quad x \in \mathbb{R}^n.$$

These estimates imply that, if $t \in [t_1, T]$,

$$\begin{aligned} |D\chi(t,x)| &\leq (C_1(T-t) + \widetilde{A})\chi(t,x)|D\psi(x)| \\ &\leq C\chi(t,x)\frac{[\psi(x)]^{\frac{1}{2}}}{(|x|^2+1)^{\frac{1}{2}}}, \end{aligned}$$

and, similarly

$$|D^2\chi(t,x)| \leq C\chi(t,x)\frac{\psi(x)}{|x|^2 + 1}.$$

We should notice that the above estimates do not depend on C_1 because of the definition of t_1. Then, since γ is bounded and since ψ is Lipschitz continuous in $\mathbb{R}^n$, we have after a long but straight-forward calculus,

$$\chi(t, x + \gamma(t,x,u,v,e)) - \chi(t,x) - D\chi(t,x).\gamma(t,x,u,v,e) \leq C\chi(t,x)\frac{\psi(x)}{|x|^2 + 1}|\gamma(t,x,u,v,e)|^2,$$

and

$$\chi(t, x + \gamma(t,x,u,v,e)) - \chi(t,x) \leq C\chi(t,x)\frac{[\psi(x)]^{\frac{1}{2}}}{(|x|^2 + 1)^{\frac{1}{2}}}|\gamma(t,x,u,v,e)|.$$

In virtue with the above estimates we have

$$\frac{\partial}{\partial t}\chi(t,x) + \sup_{u\in U, v\in V}\{A^{u,v}\chi(t,x) + B^{u,v}\chi(t,x) + K\chi(t,x)+$$
$$K|D\chi(t,x).\sigma(t,x,u,v)| + K(C^{u,v}\chi(t,x))^+\}$$
$$\leq -\chi(t,x)\{C_1\psi(x) - C\psi(x) - C[\psi(x)]^{\frac{1}{2}} - \frac{C\psi(x)}{|x|^2+1} - K - CK[\psi(x)]^{\frac{1}{2}} - CK\frac{[\psi(x)]^{\frac{1}{2}}}{(|x|^2+1)^{\frac{1}{2}}}\}$$
$$< -\chi(t,x)\{C_1 - [C+K]\}\psi(x) < 0, \text{ if } C_1 > C + K \text{ large enough.}$$

$\square$

Now we can continue to prove the uniqueness theorem—Theorem 5.1.

Proof of Theorem 5.1. (continued) Let us put $\omega := u_1 - u_2$. Then we have, for some $\widetilde{A} > 0$,

$$\lim_{|x|\to\infty} \omega(t,x)e^{-\widetilde{A}[\log((|x|^2+1)^{\frac{1}{2}})]^2} = 0,$$

uniformly with respect to $t \in [0,T]$. This implies, in particular, that for any $\alpha > 0$, $\omega(t,x) - \alpha\chi(t,x)$ is bounded from above in $[t_1,T]\times\mathbb{R}^n$, and that

$$M := \max_{[t_1,T]\times\mathbb{R}^n}(\omega - \alpha\chi)(t,x)e^{-K(T-t)}$$

is achieved at some point $(t_0,x_0) \in [t_1,T]\times\mathbb{R}^n$ (depending on α). We now have to distinguish between two cases.

For the first case we suppose that: $\omega(t_0,x_0) \leq 0$, for any $\alpha > 0$.

Then, obviously $M \leq 0$ and $u_1(t,x) - u_2(t,x) \leq \alpha\chi(t,x)$ in $[t_1,T]\times\mathbb{R}^n$. Consequently, letting α tend to zero we obtain

$$u_1(t,x) \leq u_2(t,x), \text{ for all } (t,x) \in [t_1,T]\times\mathbb{R}^n.$$

For the second case we assume that there exists some $\alpha > 0$ such that $\omega(t_0,x_0) > 0$.

We notice that $\omega(t,x) - \alpha\chi(t,x) \leq (\omega(t_0,x_0) - \alpha\chi(t_0,x_0))e^{-K(t-t_0)}$ in $[t_1,T]\times\mathbb{R}^n$. Then, putting

$$\varphi(t,x) = \alpha\chi(t,x) + (\omega - \alpha\chi)(t_0,x_0)e^{-K(t-t_0)}$$

we get $\omega - \varphi \leq 0 = (\omega - \varphi)(t_0,x_0)$ in $[t_1,T]\times\mathbb{R}^n$. Consequently, since ω is a viscosity subsolution of (5.2) from Lemma 5.1 we have

$$\frac{\partial}{\partial t}\varphi(t_0,x_0) + \sup_{u\in U, v\in V}\{A^{u,v}\varphi(t_0,x_0)) + B^{u,v}\varphi(t_0,x_0)+$$
$$K|\varphi(t_0,x_0)| + K|D\varphi(t_0,x_0).\sigma(t_0,x_0,u,v)| + K(C^{u,v}\varphi(t_0,x_0))^+\} \geq 0.$$

Moreover, due to our assumption that $\omega(t_0,x_0) > 0$ and since $\omega(t_0,x_0) = \varphi(t_0,x_0)$ we can replace $K|\varphi(t_0,x_0)|$ by $K\varphi(t_0,x_0)$ in the above formula. Then, from the definition of φ and Lemma 5.2,

$$0 \leq \alpha\{\frac{\partial\chi}{\partial t}(t_0,x_0) + \sup_{u\in U, v\in V}\{A^{u,v}\chi(t_0,x_0)) + B^{u,v}\chi(t_0,x_0)+$$
$$+K\chi(t_0,x_0) + K|D\chi(t_0,x_0).\sigma(t_0,x_0,u,v)| + K(C^{u,v}\chi(t_0,x_0))^+\}\} < 0$$

which is a contradiction. Finally, by applying successively the same argument on the interval $[t_2,t_1]$ with $t_2 = (t_1 - \frac{\widetilde{A}}{C_1})^+$, and then, if $t_2 > 0$, on $[t_3,t_2]$ with $t_3 = (t_2 - \frac{\widetilde{A}}{C_1})^+$, etc. We get

$$u_1(t,x) \leq u_2(t,x), \quad (t,x) \in [0,T]\times\mathbb{R}^n.$$

Thus, the proof is complete. $\square$

Remark 5.1. *Obviously, since the lower value function $W(t,x)$ is of at most linear growth it belongs to Θ, and so $W(t,x)$ is the unique viscosity solution in Θ of equation (5.1). Similarly we get that the upper value function $U(t,x)$ is the unique viscosity solution in Θ of equation (4.2).*

Remark 5.2. *If the Isaacs' condition holds, that is, if for all $(t, x) \in [0, T] \times \mathbb{R}^n$,*

$$H^-(t, x, \Psi(t,x), D\Psi(t,x), D^2\Psi(t,x)) = H^+(t, x, \Psi(t,x), D\Psi(t,x), D^2\Psi(t,x)),$$

then the equations (5.1) and (4.2) coincide and from the uniqueness in Θ of viscosity solution it follows that the lower value function $W(t,x)$ equals to the upper value function $U(t,x)$ which means the associated stochastic differential game has a value.

6   Appendix

6.1   FBSDEs with Jumps

In this subsection we give an overview over basic results on BSDEs with jumps associated with Forward SDEs with jumps (for short: FBSDEs) for reader's convenience. We consider measurable functions $b : [0, T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0, T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^{n \times d}$ and $\gamma : [0, T] \times \Omega \times \mathbb{R}^n \times E \to \mathbb{R}^n$ which are supposed to satisfy the following conditions:

(i) $b(\cdot, 0)$ and $\sigma(\cdot, 0)$ are $\mathcal{F}_t$-adapted processes, and there exists some
   constant $C > 0$ such that
   $$|b(t, 0)| + |\sigma(t, 0)| \leq C, \quad \text{dtdP-}a.e.;$$

(ii) b and σ are Lipschitz in x, i.e., there is some constant $C > 0$ such that
   $$|b(t, x) - b(t, x')| + |\sigma(t, x) - \sigma(t, x')| \leq C|x - x'|, \quad \text{dtdP-}a.e.,$$
   $$\text{for } \ x, \ x' \in \mathbb{R}^n; \qquad \text{(H6.1)}$$

(iii) There exists a measurable function $\rho : E \to \mathbb{R}^+$ with $\int_E \rho^2(e)\lambda(de) < +\infty$,
   such that, for any $x, y \in \mathbb{R}^n$ and $e \in E$,
   $$|\gamma(t, x, e) - \gamma(t, y, e)| \ \leq \ \rho(e)|x - y|,$$
   $$\gamma(\cdot, e) \text{ is } \mathcal{F}_t\text{-predictable, and } |\gamma(t, x, e)| \ \leq \ \rho(e)(1 + |x|), \ \text{dtdP-}a.e..$$

We now consider the following SDE with jumps parameterized by the initial condition $(t, \zeta) \in [0, T] \times L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$:

$$\begin{cases} dX_s^{t,\zeta} &= b(s, X_s^{t,\zeta})ds + \sigma(s, X_s^{t,\zeta})dB_s + \int_E \gamma(s, X_{s-}^{t,\zeta}, e)\tilde{\mu}(ds, de), \\ X_t^{t,\zeta} &= \zeta, \hspace{6cm} s \in [t, T]. \end{cases} \qquad (6.1)$$

Under the assumption (H6.1), SDE (6.1) has a unique strong solution and, there exists $C \in \mathbb{R}^+$ such that, for any $t \in [0, T]$ and $\zeta, \zeta' \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$,

$$\begin{aligned} E[\sup_{t \leq s \leq T} |X_s^{t,\zeta} - X_s^{t,\zeta'}|^2|\mathcal{F}_t] &\leq C|\zeta - \zeta'|^2, \quad a.s., \\ E[\sup_{t \leq s \leq T} |X_s^{t,\zeta}|^2|\mathcal{F}_t] &\leq C(1 + |\zeta|^2), \quad a.s. \end{aligned} \qquad (6.2)$$

(Referred to Proposition 1.1 in [1]). We emphasize that the constant C in (6.2) only depends on the Lipschitz and the growth constants of b, σ and γ.

Let now be given two real valued functions $f(t, x, y, z, k)$ and $\Phi(x)$ which shall satisfy the following

conditions:

(i) $\Phi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is an $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}^n)$-measurable random variable and

$f : [0,T] \times \Omega \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times L^2(E, \mathcal{B}(E), \lambda; \mathbb{R}) \to \mathbb{R}$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$
$\otimes \mathcal{B}(L^2(E, \mathcal{B}(E), \lambda; \mathbb{R}))$-measurable.

(ii)   There exists a constant $C > 0$ such that

$$|f(t,x,y,z,k) - f(t,x',y',z',k')| + |\Phi(x) - \Phi(x')|$$
$$\leq C(|x - x'| + |y - y'| + |z - z'| + ||k - k'||), \quad a.s., \quad \text{(H6.2)}$$

for all $0 \leq t \leq T$, x, $x' \in \mathbb{R}^n$, y, $y' \in \mathbb{R}$, z, $z' \in \mathbb{R}^d$ and k, $k' \in L^2(E, \mathcal{B}(E), \lambda; \mathbb{R})$.

(iii) f and Φ satisfy a linear growth condition, i.e., there exists some $C > 0$

such that, dt $\times$ dP-a.e., for all $x \in \mathbb{R}^n$,
$$|f(t,x,0,0,0)| + |\Phi(x)| \leq C(1 + |x|).$$

With the help of the above assumptions we can verify that the coefficient $f(s, X_s^{t,\zeta}, y, z, k)$ satisfies the hypotheses (A1) and (A2) and $\xi = \Phi(X_T^{t,\zeta}) \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R})$. Therefore, the following BSDE with jump possesses a unique solution:

$$\begin{cases} -dY_s^{t,\zeta} &= f(s, X_s^{t,\zeta}, Y_s^{t,\zeta}, Z_s^{t,\zeta}, K_s^{t,\zeta})ds - Z_s^{t,\zeta}dB_s - \int_E K_{s-}^{t,\zeta}(e)\tilde{\mu}(ds, de), \\ Y_T^{t,\zeta} &= \Phi(X_T^{t,\zeta}), \qquad\qquad\qquad\qquad\qquad\qquad s \in [t,T]. \end{cases} \quad \text{(6.3)}$$

Proposition 6.1. *We suppose that the hypotheses (H6.1) and (H6.2) hold. Then, for any $0 \leq t \leq T$ and the associated initial conditions $\zeta, \zeta' \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, we have the following estimates:*

$$(i) E[\sup_{t \leq s \leq T} |Y_s^{t,\zeta}|^2 + \int_t^T |Z_s^{t,\zeta}|^2 ds | \mathcal{F}_t] + E[\int_t^T \int_E |K_s^{t,\zeta}(e)|^2 \lambda(de) ds | \mathcal{F}_t]$$
$$\leq C(1 + |\zeta|^2), \quad a.s.;$$
$$(ii) E[\sup_{t \leq s \leq T} |Y_s^{t,\zeta} - Y_s^{t,\zeta'}|^2 + \int_t^T |Z_s^{t,\zeta} - Z_s^{t,\zeta'}|^2 ds | \mathcal{F}_t] +$$
$$E[\int_t^T \int_E |K_s^{t,\zeta}(e) - K_s^{t,\zeta'}(e)|^2 \lambda(de) ds | \mathcal{F}_t] \leq C|\zeta - \zeta'|^2, \quad a.s.$$

In particular,

$$(iii) \quad |Y_t^{t,\zeta}| \leq C(1 + |\zeta|), \quad a.s.; \quad\quad\quad\quad\quad\quad\quad\quad (6.4)$$
$$(iv) \quad |Y_t^{t,\zeta} - Y_t^{t,\zeta'}| \leq C|\zeta - \zeta'|, \quad a.s.$$

The above constant $C > 0$ depends only on the Lipschitz and the growth constants of b, σ, γ, f and Φ.

From Lemma 2.1, the estimate (6.2) and Itô's formula we can prove this proposition .

Let us now introduce the random field:

$$u(t,x) = Y_s^{t,x}|_{s=t}, \ (t,x) \in [0,T] \times \mathbb{R}^n, \quad\quad\quad\quad\quad\quad (6.5)$$

where $Y^{t,x}$ is the solution of BSDE (6.3) with $x \in \mathbb{R}^n$ at the place of $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$.

As a consequence of Proposition 6.1 we have that, for all $t \in [0,T]$, P-a.s.,

$$(i) \quad |u(t,x) - u(t,y)| \leq C|x-y|, \text{ for all } x, y \in \mathbb{R}^n;$$
$$(ii) \quad |u(t,x)| \leq C(1 + |x|), \text{ for all } x \in \mathbb{R}^n. \quad\quad\quad\quad (6.6)$$

Remark 6.1. *In the general situation u is random and forms an adapted random field, that is, for any $x \in \mathbb{R}^n$, $u(\cdot, x)$ is an $\mathcal{F}_t$−adapted real valued process. Indeed, recall that b, σ, f and Φ all are $\mathcal{F}_t$-adapted random functions. On the other hand, it is well known that, under the additional assumption that the functions*

$$b, \sigma, \gamma, f \text{ and } \Phi \text{ are deterministic,} \quad\quad\quad\quad\quad\quad (H6.3)$$

u is also a deterministic function of (t,x).

The random field u and $Y^{t,\zeta}$, $(t,\zeta) \in [0,T] \times L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, are related by the following theorem.

Theorem 6.1. *Under the assumptions (H6.1) and (H6.2), for any $t \in [0, T]$ and $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, we have*

$$u(t, \zeta) = Y_t^{t,\zeta}, \quad P\text{-}a.s.. \tag{6.7}$$

Remark 6.2. *Obviously, $Y_s^{t,\zeta} = Y_s^{s, X_s^{t,\zeta}} = u(s, X_s^{t,\zeta}), \quad P\text{-}a.s..$*

The proof of Theorem 6.1 is similar to the proof in Peng [18] for the FBSDE with Brownian motion, also can refer to Theorem A.1 in [16], for reader's convenience we give the proof here. It makes use of the following definition.

Definition 6.1. *For any $t \in [0, T]$, a sequence $\{A_i\}_{i=1}^N \subset \mathcal{F}_t$ (with $1 \leq N \leq \infty$) is called a partition of $(\Omega, \mathcal{F}_t)$ if $\cup_{i=1}^N A_i = \Omega$ and $A_i \cap A_j = \phi$, whenever $i \neq j$.*

Proof (of Theorem 6.1): We first consider the case where ζ is a simple random variable of the form

$$\zeta = \sum_{i=1}^N x_i \mathbf{1}_{A_i}, \tag{6.8}$$

where $\{A_i\}_{i=1}^N$ is a finite partition of $(\Omega, \mathcal{F}_t)$ and $x_i \in \mathbb{R}^n$, for $1 \leq i \leq N$.
For each i, we put $(X_s^i, Y_s^i, Z_s^i, K_s^i) \equiv (X_s^{t,x_i}, Y_s^{t,x_i}, Z_s^{t,x_i}, K_s^{t,x_i})$. Then X^i is the solution of the SDE

$$X_s^i = x_i + \int_t^s b(r, X_r^i)dr + \int_t^s \sigma(r, X_r^i)dB_r + \int_t^s \int_E \gamma(r, X_{r-}^i, e)\widetilde{\mu}(dr, de), \ s \in [t, T],$$

and (Y^i, Z^i, K^i) is the solution of the associated BSDE

$$Y_s^i = \Phi(X_T^i) + \int_s^T f(r, X_r^i, Y_r^i, Z_r^i, K_r^i)dr - \int_s^T Z_r^i dB_r - \int_s^T \int_E K_r^i(e)\widetilde{\mu}(dr, de), \ s \in [t, T].$$

The above two equations are multiplied by $\mathbf{1}_{A_i}$ and summed up with respect to i. Thus, taking into account that $\sum_i \varphi(x_i)\mathbf{1}_{A_i} = \varphi(\sum_i x_i \mathbf{1}_{A_i})$, we get

$$
\begin{aligned}
\sum_{i=1}^N \mathbf{1}_{A_i} X_s^i &= \zeta + \int_t^s b(r, \sum_{i=1}^N \mathbf{1}_{A_i} X_r^i)dr + \int_t^s \sigma(r, \sum_{i=1}^N \mathbf{1}_{A_i} X_r^i)dB_r \\
&\quad + \int_t^s \int_E \gamma(r, \sum_{i=1}^N \mathbf{1}_{A_i} X_{r-}^i, e)\widetilde{\mu}(dr, de);
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{i=1}^N \mathbf{1}_{A_i} Y_s^i &= \Phi(\sum_{i=1}^N \mathbf{1}_{A_i} X_T^i) + \int_s^T f(r, \sum_{i=1}^N \mathbf{1}_{A_i} X_r^i, \sum_{i=1}^N \mathbf{1}_{A_i} Y_r^i, \sum_{i=1}^N \mathbf{1}_{A_i} Z_r^i, \sum_{i=1}^N \mathbf{1}_{A_i} K_r^i)dr \\
&\quad - \int_s^T \sum_{i=1}^N \mathbf{1}_{A_i} Z_r^i dB_r - \int_s^T \int_E \sum_{i=1}^N \mathbf{1}_{A_i} K_r^i(e)\widetilde{\mu}(dr, de).
\end{aligned}
$$

Then the strong uniqueness property of the solution of the SDE and the BSDE yields

$$X_s^{t,\zeta} = \sum_{i=1}^N X_s^i \mathbf{1}_{A_i}, \ (Y_s^{t,\zeta}, Z_s^{t,\zeta}, K_s^{t,\zeta}) = (\sum_{i=1}^N \mathbf{1}_{A_i} Y_s^i, \sum_{i=1}^N \mathbf{1}_{A_i} Z_s^i, \sum_{i=1}^N \mathbf{1}_{A_i} K_s^i), \ s \in [t, T].$$

Finally, from $u(t, x_i) = Y_t^i, \ 1 \leq i \leq N$, we deduce that

$$Y_t^{t,\zeta} = \sum_{i=1}^N Y_t^i \mathbf{1}_{A_i} = \sum_{i=1}^N u(t, x_i)\mathbf{1}_{A_i} = u(t, \sum_{i=1}^N x_i \mathbf{1}_{A_i}) = u(t, \zeta).$$

Therefore, for simple random variables, we have the desired result.

Given a general $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$ we can choose a sequence of simple random variables $\{\zeta_i\}$ which converges to ζ in $L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$. Consequently, from the estimates (6.4), (6.6) and the first step of the proof, we have

$$
\begin{aligned}
E|Y_t^{t,\zeta_i} - Y_t^{t,\zeta}|^2 &\leq CE|\zeta_i - \zeta|^2 \to 0, \ i \to \infty, \\
E|u(t,\zeta_i) - u(t,\zeta)|^2 &\leq CE|\zeta_i - \zeta|^2 \to 0, \ i \to \infty, \\
\text{and} \qquad Y_t^{t,\zeta_i} &= u(t,\zeta_i), \ i \geq 1.
\end{aligned}
$$

Then the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

6.2 The Proof of Theorem 3.1

Proof. To simplify notations we put

$$
W_\delta(t,x) = \operatorname*{essinf}_{\beta \in \mathcal{B}_{t,t+\delta}} \operatorname*{esssup}_{u \in \mathcal{U}_{t,t+\delta}} G_{t,t+\delta}^{t,x;u,\beta(u)}[W(t+\delta, X_{t+\delta}^{t,x;u,\beta(u)})].
$$

In analogy to $W(t,x)$ it can be easily shown that $W_\delta(t,x)$ is well-defined. The proof that $W_\delta(t,x)$ coincides with $W(t,x)$ will be split into a sequel of lemmas which all supposed that (H3.1) and (H3.2) are satisfied.

Lemma 6.1. $W_\delta(t,x)$ *is deterministic.*

The proof of this lemma uses the same ideas as that of Proposition 3.1 so it is omitted here. $\qquad$ $\square$

Lemma 6.2. $W_\delta(t,x) \leq W(t,x)$.

Proof. Let $\beta \in \mathcal{B}_{t,T}$ be arbitrarily fixed. Then, given a $u_2(\cdot) \in \mathcal{U}_{t+\delta,T}$, we define as follows the restriction β_1 of β to $\mathcal{U}_{t+\delta,T}$:

$$
\beta_1(u_1) := \beta(u_1 \oplus u_2)|_{[t,t+\delta]}, \quad u_1(\cdot) \in \mathcal{U}_{t,t+\delta},
$$

where $u_1 \oplus u_2 := u_1 \mathbf{1}_{[t,t+\delta]} + u_2 \mathbf{1}_{(t+\delta,T]}$ extends $u_1(\cdot)$ to an element of $\mathcal{U}_{t,T}$. It is easy to check that $\beta_1 \in \mathcal{B}_{t,t+\delta}$. Moreover, from the nonanticipativity property of β we deduce that β_1 is independent of the special choice of $u_2(\cdot) \in \mathcal{U}_{t+\delta,T}$. Consequently, from the definition of $W_\delta(t,x)$,

$$
W_\delta(t,x) \leq \operatorname*{esssup}_{u_1 \in \mathcal{U}_{t,t+\delta}} G_{t,t+\delta}^{t,x;u_1,\beta_1(u_1)}[W(t+\delta, X_{t+\delta}^{t,x;u_1,\beta_1(u_1)})], \text{ P-a.s.} \tag{6.9}
$$

We use the notation $I_\delta(t,x,u,v) := G_{t,t+\delta}^{t,x;u,v}[W(t+\delta, X_{t+\delta}^{t,x;u,v})]$ and notice that there exists a sequence $\{u_i^1, \ i \geq 1\} \subset \mathcal{U}_{t,t+\delta}$ such that

$$
I_\delta(t,x,\beta_1) := \operatorname*{esssup}_{u_1 \in \mathcal{U}_{t,t+\delta}} I_\delta(t,x,u_1,\beta_1(u_1)) = \sup_{i \geq 1} I_\delta(t,x,u_i^1,\beta_1(u_i^1)), \quad \text{P-a.s..}
$$

For any $\varepsilon > 0$, we put $\widetilde{\Gamma}_i := \{I_\delta(t,x,\beta_1) \leq I_\delta(t,x,u_i^1,\beta_1(u_i^1)) + \varepsilon\} \in \mathcal{F}_t, \ i \geq 1$. Then $\Gamma_1 := \widetilde{\Gamma}_1, \ \Gamma_i := \widetilde{\Gamma}_i \backslash (\cup_{l=1}^{i-1} \widetilde{\Gamma}_l) \in \mathcal{F}_t, \ i \geq 2$, form an $(\Omega, \mathcal{F}_t)$-partition, and $u_1^\varepsilon := \sum_{i \geq 1} \mathbf{1}_{\Gamma_i} u_i^1$ belongs obviously to $\mathcal{U}_{t,t+\delta}$. Moreover, from the nonanticipativity of β_1 we have $\beta_1(u_1^\varepsilon) = \sum_{i \geq 1} \mathbf{1}_{\Gamma_i} \beta_1(u_i^1)$, and from the uniqueness of the solution of the FBSDE, we deduce that $I_\delta(t,x,u_1^\varepsilon,\beta_1(u_1^\varepsilon)) = \sum_{i \geq 1} \mathbf{1}_{\Gamma_i} I_\delta(t,x,u_i^1,\beta_1(u_i^1))$, P-a.s. Hence,

$$
\begin{aligned}
W_\delta(t,x) \leq I_\delta(t,x,\beta_1) &\leq \sum_{i \geq 1} \mathbf{1}_{\Gamma_i} I_\delta(t,x,u_i^1,\beta_1(u_i^1)) + \varepsilon = I_\delta(t,x,u_1^\varepsilon,\beta_1(u_1^\varepsilon)) + \varepsilon \\
&= G_{t,t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}[W(t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)})] + \varepsilon, \text{ P-a.s.}
\end{aligned} \tag{6.10}
$$

On the other hand, using the fact that $\beta_1(\cdot) := \beta(\cdot \oplus u_2) \in \mathcal{B}_{t,t+\delta}$ does not depend on $u_2(\cdot) \in \mathcal{U}_{t+\delta,T}$ we can define $\beta_2(u_2) := \beta(u_1^\varepsilon \oplus u_2)|_{[t+\delta,T]}$, for all $u_2(\cdot) \in \mathcal{U}_{t+\delta,T}$. The such defined $\beta_2 : \mathcal{U}_{t+\delta,T} \to \mathcal{V}_{t+\delta,T}$ belongs to $\mathcal{B}_{t+\delta,T}$ since $\beta \in \mathcal{B}_{t,T}$. Therefore, from the definition of $W(t+\delta,y)$ we have, for any $y \in \mathbb{R}^n$,

$$
W(t+\delta,y) \leq \operatorname*{esssup}_{u_2 \in \mathcal{U}_{t+\delta,T}} J(t+\delta,y;u_2,\beta_2(u_2)), \text{ P-a.s.}
$$

Finally, because there exists a constant $C \in \mathbb{R}$ such that

$$
\begin{aligned}
&\text{(i)} \quad |W(t+\delta,y) - W(t+\delta,y')| \leq C|y-y'|, \text{ for any } y, \ y' \in \mathbb{R}^n; \\
&\text{(ii)} \quad |J(t+\delta,y,u_2,\beta_2(u_2)) - J(t+\delta,y',u_2,\beta_2(u_2))| \leq C|y-y'|, \text{ P-a.s.,} \\
&\qquad\qquad \text{for any } u_2 \in \mathcal{U}_{t+\delta,T},
\end{aligned} \tag{6.11}
$$

(see Lemma 3.3-(i) and (3.6)-(i)) we can show by approximating $X_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}$ that

$$W(t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}) \le \mathrm{esssup}_{u_2 \in \mathcal{U}_{t+\delta,T}} J(t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_2, \beta_2(u_2)), \ \text{P-a.s.}$$

To estimate the right side of the latter inequality we note that there exists some sequence $\{u_j^2, \ j \ge 1\} \subset \mathcal{U}_{t+\delta,T}$ such that

$$\mathrm{esssup}_{u_2 \in \mathcal{U}_{t+\delta,T}} J(t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_2, \beta_2(u_2)) = \sup_{j\ge 1} J(t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_j^2, \beta_2(u_j^2)), \ \text{P-a.s.}$$

Then, putting
$\widetilde{\Delta}_j := \{\mathrm{esssup}_{u_2 \in \mathcal{U}_{t+\delta,T}} J(t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_2, \beta_2(u_2)) \le J(t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_j^2, \beta_2(u_j^2)) + \varepsilon\} \in \mathcal{F}_{t+\delta}, \ j \ge 1$; we have with $\Delta_1 := \widetilde{\Delta}_1$, $\Delta_j := \widetilde{\Delta}_j \backslash (\cup_{l=1}^{j-1} \widetilde{\Delta}_l) \in \mathcal{F}_{t+\delta}, \ j \ge 2$, an $(\Omega, \mathcal{F}_{t+\delta})$-partition and $u_2^\varepsilon := \sum_{j\ge 1} \mathbf{1}_{\Delta_j} u_j^2 \in \mathcal{U}_{t+\delta,T}$. From the nonanticipativity of β_2 we have $\beta_2(u_2^\varepsilon) = \sum_{j\ge 1} \mathbf{1}_{\Delta_j} \beta_2(u_j^2)$, and from the definition of $\beta_1, \ \beta_2$ we know that $\beta(u_1^\varepsilon \oplus u_2^\varepsilon) = \beta_1(u_1^\varepsilon) \oplus \beta_2(u_2^\varepsilon)$. Thus, again from the uniqueness of the solution of our FBSDE, we get

$$
\begin{aligned}
J(t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_2^\varepsilon, \beta_2(u_2^\varepsilon)) &= Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_2^\varepsilon, \beta_2(u_2^\varepsilon)} && \text{(see (3.8))} \\
&= \sum_{j\ge 1} \mathbf{1}_{\Delta_j} Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_j^2, \beta_2(u_j^2)} \\
&= \sum_{j\ge 1} \mathbf{1}_{\Delta_j} J(t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_j^2, \beta_2(u_j^2)), \ \text{P-a.s.}
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
W(t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}) &\le \mathrm{esssup}_{u_2 \in \mathcal{U}_{t+\delta,T}} J(t+\delta, X_{t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_2, \beta_2(u_2)) \\
&\le \sum_{j\ge 1} \mathbf{1}_{\Delta_j} Y_{t+\delta}^{t,x;u_1^\varepsilon \oplus u_j^2, \beta(u_1^\varepsilon \oplus u_j^2)} + \varepsilon \\
&= Y_{t+\delta}^{t,x;u_1^\varepsilon \oplus u_2^\varepsilon, \beta(u_1^\varepsilon \oplus u_2^\varepsilon)} + \varepsilon \\
&= Y_{t+\delta}^{t,x;u^\varepsilon, \beta(u^\varepsilon)} + \varepsilon, \ \text{P-a.s.},
\end{aligned}
\tag{6.12}
$$

where $u^\varepsilon := u_1^\varepsilon \oplus u_2^\varepsilon \in \mathcal{U}_{t,T}$. From (6.10), (6.12), Lemma 2.2 (comparison theorem for BSDEs) and Lemma 2.3 we have, for some constant $C \in \mathbb{R}$,

$$
\begin{aligned}
W_\delta(t,x) &\le G_{t,t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}[Y_{t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)} + \varepsilon] + \varepsilon \\
&\le G_{t,t+\delta}^{t,x;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}[Y_{t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}] + (C+1)\varepsilon \\
&= G_{t,t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}[Y_{t+\delta}^{t,x;u^\varepsilon,\beta(u^\varepsilon)}] + (C+1)\varepsilon \\
&= Y_t^{t,x;u^\varepsilon,\beta(u^\varepsilon)} + (C+1)\varepsilon \\
&\le \mathrm{esssup}_{u \in \mathcal{U}_{t,T}} Y_t^{t,x;u,\beta(u)} + (C+1)\varepsilon, \ \text{P-a.s.}
\end{aligned}
\tag{6.13}
$$

Since $\beta \in \mathcal{B}_{t,T}$ has been arbitrarily chosen we have (6.13) for all $\beta \in \mathcal{B}_{t,T}$. Therefore,

$$W_\delta(t,x) \le \mathrm{essinf}_{\beta \in \mathcal{B}_{t,T}} \mathrm{esssup}_{u \in \mathcal{U}_{t,T}} Y_t^{t,x;u,\beta(u)} + (C+1)\varepsilon = W(t,x) + (C+1)\varepsilon. \tag{6.14}$$

Finally, letting $\varepsilon \downarrow 0$, we get $W_\delta(t,x) \le W(t,x)$. $\qquad \square$

Lemma 6.3. $W(t,x) \le W_\delta(t,x)$.

Proof. We continue to use the notations introduced above. From the definition of $W_\delta(t,x)$ we have

$$
\begin{aligned}
W_\delta(t,x) &= \mathrm{essinf}_{\beta_1 \in \mathcal{B}_{t,t+\delta}} \mathrm{esssup}_{u_1 \in \mathcal{U}_{t,t+\delta}} G_{t,t+\delta}^{t,x;u_1,\beta_1(u_1)}[W(t+\delta, X_{t+\delta}^{t,x;u_1,\beta_1(u_1)})] \\
&= \mathrm{essinf}_{\beta_1 \in \mathcal{B}_{t,t+\delta}} I_\delta(t,x,\beta_1),
\end{aligned}
$$

and, for some sequence $\{\beta_i^1, \ i \ge 1\} \subset \mathcal{B}_{t,t+\delta}$,

$$W_\delta(t,x) = \inf_{i\ge 1} I_\delta(t,x,\beta_i^1), \ \text{P-a.s.}$$

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For any $\varepsilon > 0$, we let $\widetilde{\Lambda}_i := \{I_\delta(t,x,\beta_i^1) - \varepsilon \le W_\delta(t,x)\} \in \mathcal{F}_t$, $i \ge 1$, $\Lambda_1 := \widetilde{\Lambda}_1$ and $\Lambda_i := \widetilde{\Lambda}_i \backslash (\cup_{l=1}^{i-1} \widetilde{\Lambda}_l) \in \mathcal{F}_t$, $i \ge 2$. Then $\{\Lambda_i, \ i \ge 1\}$ is an $(\Omega, \mathcal{F}_t)$-partition, $\beta_1^\varepsilon := \sum_{i \ge 1} \mathbf{1}_{\Lambda_i} \beta_i^1$ belongs to $\mathcal{B}_{t,t+\delta}$, and from the uniqueness of the solution of our FBSDE we conclude that $I_\delta(t,x,u_1,\beta_1^\varepsilon(u_1)) = \sum_{i \ge 1} \mathbf{1}_{\Lambda_i} I_\delta(t,x,u_1,\beta_i^1(u_1))$, P-a.s., for all $u_1(\cdot) \in \mathcal{U}_{t,t+\delta}$. Hence,

$$
\begin{aligned}
W_\delta(t,x) &\ge \sum_{i \ge 1} \mathbf{1}_{\Lambda_i} I_\delta(t,x,\beta_i^1) - \varepsilon \\
&\ge \sum_{i \ge 1} \mathbf{1}_{\Lambda_i} I_\delta(t,x,u_1,\beta_i^1(u_1)) - \varepsilon \\
&= I_\delta(t,x,u_1,\beta_1^\varepsilon(u_1)) - \varepsilon \\
&= G_{t,t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}[W(t+\delta, X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)})] - \varepsilon, \ \text{P-a.s., for all} \ u_1 \in \mathcal{U}_{t,t+\delta}.
\end{aligned}
\tag{6.15}
$$

On the other hand, from the definition of $W(t+\delta, y)$, with the same technique as before, we deduce that, for any $y \in \mathbb{R}^n$, there exists $\beta_y^\varepsilon \in \mathcal{B}_{t+\delta,T}$ such that

$$
W(t+\delta, y) \ge \operatorname{esssup}_{u_2 \in \mathcal{U}_{t+\delta,T}} J(t+\delta, y; u_2, \beta_y^\varepsilon(u_2)) - \varepsilon, \ \text{P-a.s.}
\tag{6.16}
$$

Let $\{O_i\}_{i \ge 1} \subset \mathcal{B}(\mathbb{R}^n)$ be a decomposition of $\mathbb{R}^n$ such that $\sum_{i \ge 1} O_i = \mathbb{R}^n$ and $\operatorname{diam}(O_i) \le \varepsilon$, $i \ge 1$. Moreover, we fix arbitrarily for each $i \ge 1$ an element y_i of O_i, $i \ge 1$. Then, defining $[X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}] := \sum_{i \ge 1} y_i \mathbf{1}_{\{X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)} \in O_i\}}$, we have

$$
|X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)} - [X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}]| \le \varepsilon, \ \text{everywhere on} \ \Omega, \ \text{for all} \ u_1 \in \mathcal{U}_{t,t+\delta}.
\tag{6.17}
$$

Furthermore, as we have seen above, for each y_i there exists some $\beta_{y_i}^\varepsilon \in \mathcal{B}_{t+\delta,T}$ such that (6.16) holds, and, clearly, $\beta_{u_1}^\varepsilon := \sum_{i \ge 1} \mathbf{1}_{\{X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)} \in O_i\}} \beta_{y_i}^\varepsilon \in \mathcal{B}_{t+\delta,T}$.

Now we can define the new strategy $\beta^\varepsilon(u) := \beta_1^\varepsilon(u_1) \oplus \beta_{u_1}^\varepsilon(u_2)$, $u \in \mathcal{U}_{t,T}$, where $u_1 = u|_{[t,t+\delta]}$, $u_2 = u|_{(t+\delta,T]}$ (restriction of u to $[t,t+\delta] \times \Omega$ and $(t+\delta,T] \times \Omega$, resp.). Obviously, β^ε maps $\mathcal{U}_{t,T}$ into $\mathcal{V}_{t,T}$. Moreover, β^ε is nonanticipating: Indeed, let $S : \Omega \longrightarrow [t,T]$ be an $\mathcal{F}_r$-stopping time and $u, u' \in \mathcal{U}_{t,T}$ be such that $u \equiv u'$ on $[\![t,S]\!]$. Decomposing u, u' into $u_1, u_1' \in \mathcal{U}_{t,t+\delta}$, $u_2, u_2' \in \mathcal{U}_{t+\delta,T}$ such that $u = u_1 \oplus u_2$ and $u' = u_1' \oplus u_2'$, we have $u_1 \equiv u_1'$ on $[\![t, S \wedge (t+\delta)]\!]$ from which we get $\beta_1^\varepsilon(u_1) \equiv \beta_1^\varepsilon(u_1')$ on $[\![t, S \wedge (t+\delta)]\!]$ (recall that β_1^ε is nonanticipating). On the other hand, $u_2 \equiv u_2'$ on $]\!]t+\delta, S \vee (t+\delta)]\!] (\subset (t+\delta,T] \times \{S > t+\delta\})$, and on $\{S > t+\delta\}$ we have $X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)} = X_{t+\delta}^{t,x;u_1',\beta_1^\varepsilon(u_1')}$. Consequently, from our definition, $\beta_{u_1}^\varepsilon = \beta_{u_1'}^\varepsilon$ on $\{S > t+\delta\}$ and $\beta_{u_1}^\varepsilon(u_2) \equiv \beta_{u_1'}^\varepsilon(u_2')$ on $]\!]t+\delta, S \vee (t+\delta)]\!]$. This yields $\beta^\varepsilon(u) = \beta_1^\varepsilon(u_1) \oplus \beta_{u_1}^\varepsilon(u_2) \equiv \beta_1^\varepsilon(u_1') \oplus \beta_{u_1'}^\varepsilon(u_2') = \beta^\varepsilon(u')$ on $[\![t,S]\!]$, from where it follows that $\beta^\varepsilon \in \mathcal{B}_{t,T}$.

Let now $u \in \mathcal{U}_{t,T}$ be arbitrarily chosen and decomposed into $u_1 = u|_{[t,t+\delta]} \in \mathcal{U}_{t,t+\delta}$ and $u_2 = u|_{(t+\delta,T]} \in \mathcal{U}_{t+\delta,T}$. Then, from (6.15), (6.11)-(i), (6.17) and Lemmas 2.2 (comparison theorem) and 2.3 we obtain,

$$
\begin{aligned}
W_\delta(t,x) &\ge G_{t,t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}[W(t+\delta, X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)})] - \varepsilon \\
&\ge G_{t,t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}[W(t+\delta, [X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}]) - C\varepsilon] - \varepsilon \\
&\ge G_{t,t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}[W(t+\delta, [X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}])] - C\varepsilon \\
&= G_{t,t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}[\sum_{i \ge 1} \mathbf{1}_{\{X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)} \in O_i\}} W(t+\delta, y_i)] - C\varepsilon, \ \text{P-a.s.}
\end{aligned}
\tag{6.18}
$$

Furthermore, from (6.18), (6.11)-(ii), (6.16) and Lemmas 2.2 (comparison theorem) and 2.3, we have,

$$
\begin{aligned}
W_\delta(t,x) &\ge G_{t,t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}[\sum_{i \ge 1} \mathbf{1}_{\{X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)} \in O_i\}} J(t+\delta, y_i; u_2, \beta_{y_i}^\varepsilon(u_2)) - \varepsilon] - C\varepsilon \\
&\ge G_{t,t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}[\sum_{i \ge 1} \mathbf{1}_{\{X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)} \in O_i\}} J(t+\delta, y_i; u_2, \beta_{y_i}^\varepsilon(u_2))] - C\varepsilon \\
&= G_{t,t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}[J(t+\delta, [X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}]; u_2, \beta_{u_1}^\varepsilon(u_2))] - C\varepsilon \\
&\ge G_{t,t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}[J(t+\delta, X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}; u_2, \beta_{u_1}^\varepsilon(u_2)) - C\varepsilon] - C\varepsilon \\
&\ge G_{t,t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}[J(t+\delta, X_{t+\delta}^{t,x;u_1,\beta_1^\varepsilon(u_1)}; u_2, \beta_{u_1}^\varepsilon(u_2))] - C\varepsilon \\
&= G_{t,t+\delta}^{t,x;u,\beta^\varepsilon(u)}[Y_{t+\delta}^{t,x,u,\beta^\varepsilon(u)}] - C\varepsilon \\
&= Y_t^{t,x;u,\beta^\varepsilon(u)} - C\varepsilon, \ \text{P-a.s., for any} \ u \in \mathcal{U}_{t,T}.
\end{aligned}
\tag{6.19}
$$

Consequently,

$$
\begin{aligned}
W_\delta(t,x) &\geq \mathrm{esssup}_{u\in\mathcal{U}_{t,T}} J(t,x;u,\beta^\varepsilon(u)) - C\varepsilon \\
&\geq \mathrm{essinf}_{\beta\in\mathcal{B}_{t,T}}\mathrm{esssup}_{u\in\mathcal{U}_{t,T}} J(t,x;u,\beta(u)) - C\varepsilon \qquad (6.20) \\
&= W(t,x) - C\varepsilon, \ \text{P-a.s.}
\end{aligned}
$$

Finally, letting $\varepsilon \downarrow 0$ we get $W_\delta(t,x) \geq W(t,x)$. The proof is complete. $\qquad\square$

Remark 6.3. (i) *From the inequalities (6.10) and (6.15) we see that for all $(t,x) \in [0,T] \times \mathbb{R}^n$, $\delta > 0$ with $\delta \leq T - t$ and $\varepsilon > 0$, it holds:*
a) For every $\beta \in \mathcal{B}_{t,t+\delta}$, there exists some $u^\varepsilon(\cdot) \in \mathcal{U}_{t,t+\delta}$ such that

$$
W(t,x)(= W_\delta(t,x)) \leq G^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_{t,t+\delta}[W(t+\delta, X^{t,x;u^\varepsilon,\beta(u^\varepsilon)}_{t+\delta})] + \varepsilon, \ \ P\text{-}a.s. \qquad (6.21)
$$

b) There exists some $\beta^\varepsilon \in \mathcal{B}_{t,t+\delta}$ such that, for all $u \in \mathcal{U}_{t,t+\delta}$,

$$
W(t,x)(= W_\delta(t,x)) \geq G^{t,x;u,\beta^\varepsilon(u)}_{t,t+\delta}[W(t+\delta, X^{t,x;u,\beta^\varepsilon(u)}_{t+\delta})] - \varepsilon, \ \ P\text{-}a.s. \qquad (6.22)
$$

(ii) *Recall that the lower value function W is deterministic. Thus, by choosing $\delta = T - t$ and taking the expectation on both sides of (6.21) and (6.22) we can show that*

$$
W(t,x) = \inf_{\beta\in\mathcal{B}_{t,T}} \sup_{u\in\mathcal{U}_{t,T}} E[J(t,x;u,\beta(u))].
$$

In analogy we also have

$$
U(t,x) = \sup_{\alpha\in\mathcal{A}_{t,T}} \inf_{v\in\mathcal{V}_{t,T}} E[J(t,x;\alpha(v),v)].
$$

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